Linear Algebra and its Applications 519 (2017) 27–53

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the decay of the off-diagonal singular values in cyclic reduction ☆

Dario A. Bini [a], Stefano Massei [b], Leonardo Robol [c,*]

[a] Dipartimento di Matematica, Unversità di Pisa, L.go B. Pontecorvo, 5, 56127 Pisa, Italy
[b] Scuola Normale Superiore di Pisa, P.za Cavalieri, 7, 56126, Pisa, Italy
[c] KU Leuven, Dept. Computerwetenschappen (Postbus 02402), Celestijnenlaan 200A, 3001 Heverlee (Leuven), Belgium

A R T I C L E   I N F O

Article history:
Received 20 August 2016
Accepted 20 December 2016
Available online 27 December 2016
Submitted by M. Benzi

MSC:
41A20
60J22
65F05

Keywords:
Cyclic reduction
Quasiseparable matrices
Rational interpolation
Sylvester equations
Exponential decay
Block tridiagonal systems

A B S T R A C T

It was recently observed in [10] that the singular values of the off-diagonal blocks of the matrix sequences generated by the Cyclic Reduction algorithm decay exponentially. This property was used to solve, with a higher efficiency, certain quadratic matrix equations encountered in the analysis of queuing models. In this paper, we provide a theoretical bound to the basis of this exponential decay together with a tool for its estimation based on a rational interpolation problem. Numerical experiments show that the bound is often accurate in practice. Applications to solving $n \times n$ block tridiagonal block Toeplitz systems with $n \times n$ quasiseparable blocks and certain generalized Sylvester equations in $O(n^2 \log n)$ arithmetic operations are shown.

☆ This work has been partially supported by an INdAM/GNCS Research Project 2016, and by the Research Council KU Leuven, project CREA/13/012, and by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office, Belgian Network DYSCO.

* Corresponding author.
E-mail addresses: bini@dm.unipi.it (D.A. Bini), stefano.massei@sns.it (S. Massei), leonardo.robol@cs.kuleuven.be (L. Robol).

http://dx.doi.org/10.1016/j.laa.2016.12.027
0024-3795/

1. Introduction

Cyclic reduction, CR for short, is an algorithm originally introduced by G.H. Golub and R.W. Hockney in [18] for the solution of certain block tridiagonal linear systems coming from the finite difference discretization of elliptic PDEs. It has been later generalized and extended to other contexts, like for instance to the solution of polynomial matrix equations, and has been proven to be a successful method for solving a large class of queuing problems and infinite Markov Chains. We refer the reader to the books [9,8] and to the survey paper [11] for more details and for the many references to the literature.

Given three $m \times m$ matrices A_{-1}, A_0, A_1, and a positive integer n consider the block tridiagonal block Toeplitz matrix $\mathcal{A}_n = \mathrm{trid}_n(A_{-1}, A_0, A_1)$ having block-size n where A_0 is on the main diagonal while A_{-1} is in the lower diagonal and A_1 in the upper diagonal. For a vector $b \in \mathbb{R}^{mn}$, consider the system $\mathcal{A}_n x = b$. Roughly speaking, CR generates three sequences of $m \times m$ matrices $A_{-1}^{(h)}$, $A_0^{(h)}$ and $A_1^{(h)}$, for $h = 0, 1, \dots$, with $A_i^{(0)} = A_i$, $i = -1, 0, 1$, and a sequence of systems $\mathcal{A}_{n_h} x^{(h)} = b^{(h)}$, $\mathcal{A}_{n_h} = \mathrm{trid}_{n_h}(A_{-1}^{(h)}, A_0^{(h)}, A_1^{(h)})$, where $n_h = \lfloor n_{h-1}/2 \rfloor$ and $x^{(h)}$ is a subvector of x. This way, solving a block tridiagonal block-Toeplitz system of block size n is reduced to solving a block tridiagonal block Toeplitz system of size $\lfloor n/2 \rfloor$. The computation of $A_i^{(h)}$ given $A_i^{(h-1)}$, for $i = -1, 0, 1$, amounts to perform one matrix inversion and few matrix multiplications for the overall cost per step of $O(m^3)$ arithmetic operations (ops).

Under certain assumptions, customarily verified in many applications, the sequences $A_1^{(h)}$ and/or $A_{-1}^{(h)}$ converge doubly exponentially to zero. This makes CR a powerful tool for solving large or even infinite systems, as well as matrix equations of the kind $A_{-1} + A_0 X + A_1 X^2 = 0$, typically encountered in the analysis of queuing problems [22], where the unknown is the $m \times m$ matrix X and a solution of spectral radius at most 1 is sought.

In short, the three sequences $A_i^{(h)}$, $i = -1, 0, 1$, which are related to the Schur complements of certain principal submatrices of $\mathcal{A}_n$, are given by the following matrix recurrences where we report also two additional auxiliary sequences, namely $\widetilde{A}^{(h)}$ and $\widehat{A}^{(h)}$, which have a role in the solution of quadratic matrix equations and of linear systems when n is not of the kind $2^q - 1$:

$$
\begin{aligned}
&A_0^{(h+1)} = A_0^{(h)} - A_1^{(h)} S^{(h)} A_{-1}^{(h)} - A_{-1}^{(h)} S^{(h)} A_1^{(h)}, \quad S^{(h)} = (A_0^{(h)})^{-1} \\
&A_1^{(h+1)} = -A_1^{(h)} S^{(h)} A_1^{(h)}, \quad A_{-1}^{(h+1)} = -A_{-1}^{(h)} S^{(h)} A_{-1}^{(h)}, \qquad h = 0, 1, \dots \\
&\widehat{A}^{(h+1)} = \widehat{A}^{(h)} - A_1^{(h)} S^{(h)} A_{-1}^{(h)}, \quad \widetilde{A}^{(h+1)} = \widetilde{A}^{(h)} - A_{-1} S^{(h)} A_1^{(h)}
\end{aligned}
\tag{1}
$$

with $A_0^{(0)} = \widetilde{A}^{(0)} = \widehat{A}^{(0)} = A_0$, $A_1^{(0)} = A_1$, $A_{-1}^{(0)} = A_{-1}$.

Here we assume that all the matrices $A_0^{(h)}$ generated by the recursion are invertible so that CR can be carried out with no breakdown. This assumption is generally satisfied in the applications.