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Additive decomposability preservers and related results on tensor products of matrices

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ABSTRACT

In this paper we characterize additive maps between tensor spaces that send decomposable tensors to decomposable tensors. As an application, we classify all additive maps from tensor products of spaces of rectangular matrices to spaces of rectangular matrices which do not increase the rank of tensor product of rank one matrices.

1. Introduction and statement of main result

Motivated by the work of Frobenius [1] concerning linear preservers of the determinant of complex matrices, Dieudonné [2] characterized invertible linear maps on spaces of square matrices over an arbitrary field that send singular matrices to singular matrices.

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His result gives immediately the structure of invertible linear maps that preserve rank one matrices. In [3] Marcus and Moyls classified linear rank one preservers on rectangular matrices over an algebraically closed field with characteristics zero. Westwick [4] gave a substantial generalization of their result by establishing a general decomposition theorem for linear maps from one tensor space to another that carry non-zero decomposable tensors to non-zero decomposable tensors. He obtained this structure theorem by first proving a combinatorial result concerning maps from one Cartesian product of sets to another that send adjacent tuples to adjacent tuples. Later he was able to improve his result in [5] by characterizing linear maps from r-fold tensor spaces to s-fold tensor spaces that send decomposable tensors to decomposable tensors over a field with at least five elements. In a later short paper [6], Westwick was able to remove the restriction on the cardinality of the underlying field via field extensions. The special case where $r = s = 2$ was obtained much earlier by Lim [7]. It is worth pointing out that the proof of Westwick's structure theorem [6] is in fact also valid for infinite dimensional spaces.

In the present paper, we use Westwick's structure theorem in [6], the method of field extensions and group homomorphisms to characterize additive maps between tensor spaces over an arbitrary field that send decomposable tensors to decomposable tensors. This result generalizes previous work done by several authors concerning additive preservers of matrices (operators) of rank less than or equal to one (see [8–11]) and additive preservers of non-zero decomposable tensors [12]. As an application of our main result, we classify additive maps from tensor products of spaces of rectangular matrices to spaces of rectangular matrices over an arbitrary field that do not increase the rank of tensor product of rank one matrices. This generalizes the main result in [13,14]. We also study linear preservers of tensor products of matrices of bounded rank by first establishing a basic result concerning the Zariski closure of tensor products of homogeneous sets of matrices over an infinite field. We remark that Huang et al. [15] also obtain the structure of linear maps on tensor products of spaces of square matrices that preserve the rank of tensor products of rank one matrices by using Westwick's structure theorem [4] concerning linear preservers of non-zero decomposable tensors.

Set $[\![n]\!] = \{1, \dots, n\}$. A tensor space is a vector space U over a field $\mathbb{F}$ which is isomorphic to some tensor products of vector spaces $U_1, \dots, U_n$ over $\mathbb{F}$ with $\dim_{\mathbb{F}}(U_i) \geqslant 2$. If U is a tensor space over $\mathbb{F}$, choose a fixed representation $U_1, \dots, U_n$ and write $U = \bigotimes_{\mathbb{F}}\{U_i \mid i \in [\![n]\!]\}$. Denote by $D(U)$ the set of all elements of the forms $\otimes_{\mathbb{F}}\{u_i \mid i \in [\![n]\!]\} := u_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} u_n$ under the fixed representation. The elements of $D(U)$ are called decomposable tensors. It shall be understood that $D(U)$ is the set of all decomposable tensors of U under the representation $U_1, \dots, U_n$ when we write $U = \bigotimes_{\mathbb{F}}\{U_i \mid i \in [\![n]\!]\}$. Let R be a ring and $\mathrm{Hom}_R(U, V)$ denote the set of all R-module homomorphisms between R-modules U, V.

Let $U = \bigotimes_{\mathbb{F}}\{U_i \mid i \in [\![n]\!]\}$ and $V = \bigotimes_{\mathbb{K}}\{V_i \mid i \in [\![m]\!]\}$ be tensor spaces and $\mathbb{Z}$ be the ring of integers. It is clear that $\mathrm{Hom}_{\mathbb{Z}}(U, V)$ coincides with the set of all additive maps from U to V. A map $T \in \mathrm{Hom}_{\mathbb{Z}}(U, V)$ is called decomposable if $T(D(U)) \subseteq D(V)$. A map $T \in \mathrm{Hom}_{\mathbb{Z}}(U_i, V_j)$ is called σ-quasilinear provided $T(\alpha X) = \sigma(\alpha)T(X)$ for all