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Leonard triples extended from a given totally almost bipartite Leonard pair of Bannai/Ito type

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ABSTRACT

Let $\mathbb{K}$ denote a field of characteristic zero and let d denote an integer at least 3. Let

$$\mathbf{A} = (-1)^d \begin{pmatrix} 0 & 2d+1 & & & & & & 0 \\ 1 & 0 & 2d & & & & & \\ & 2 & 0 & 2d-1 & & & & \\ & & 3 & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & d+3 & \\ & & & & & d-1 & 0 & d+2 \\ 0 & & & & & & d & d+1 \end{pmatrix}$$

and

$$\mathbf{A}^* = diag((-1)^d(2d+1), \ldots, -7, 5, -3, 1)$$

be two matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Then $\mathbf{A}, \mathbf{A}^*$ is a totally almost bipartite Leonard pair on $\mathbb{K}^{d+1}$ of Bannai/Ito type. In this paper, we determine all the matrices $\mathbf{A}^\varepsilon \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $\mathbf{A}, \mathbf{A}^*, \mathbf{A}^\varepsilon$ form a Leonard triple on $\mathbb{K}^{d+1}$.

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1. Introduction

Throughout this paper we adopt the following notation and terminology. Let $\mathbb{K}$ denote a field of characteristic zero. Let d denote a nonnegative integer. Let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \dots, d$. Let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space consisting of all $d+1$ by 1 matrices that have entries in $\mathbb{K}$. Its rows are indexed by $0, 1, \dots, d$. We view $\mathbb{K}^{d+1}$ as a left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$.

A square matrix X is said to be *tridiagonal* whenever every nonzero entry appears on, immediately above, or immediately below the main diagonal. Assume X is tridiagonal. Then X is said to be *irreducible* whenever all entries immediately above and below the main diagonal are nonzero.

The notion of a Leonard pair was introduced by Terwilliger in [10].

Definition 1.1. ([10, Definition 1.1]) Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy both (i) and (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

Terwilliger classified the Leonard pairs up to isomorphism in [13]. By that classification, the isomorphism classes of Leonard pairs fall naturally into thirteen families: q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, affine q-Krawtchouk, quantum q-Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk, Bannai/Ito and orphan.

Lemma 1.2. *([10, Theorem 7.3]) Let A, A^* denote a Leonard pair on V. Let W denote a nonzero subspace of V with $AW \subseteq W$ and $A^*W \subseteq W$. Then $W = V$.*

Let $B \in \mathrm{Mat}_{d+1}(\mathbb{K})$ be tridiagonal. We say that B is *bipartite* whenever $B_{ii} = 0$ for $0 \le i \le d$. We say that B is *almost bipartite* whenever exactly one of $B_{0,0}, B_{d,d}$ is nonzero and $B_{ii} = 0$ for $1 \le i \le d-1$.

Definition 1.3. ([2, Definition 1.2, Definition 1.3]) A Leonard pair A, A^* is said to be *bipartite* (resp. *almost bipartite*) whenever the matrix representing A from Definition 1.1(ii) is bipartite (resp. almost bipartite). The Leonard pair A, A^* is said to be *dual bipartite* (resp. *dual almost bipartite*) whenever the Leonard pair A^*, A is bipartite (resp. almost bipartite). The Leonard pair A, A^* is said to be *totally bipartite* (resp. *totally almost bipartite*) whenever it is bipartite (resp. almost bipartite) and dual bipartite (resp. dual almost bipartite).