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Spectrally arbitrary zero–nonzero patterns and field extensions

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ABSTRACT

An $n \times n$ matrix pattern is said to be spectrally arbitrary over a field $\mathbb{F}$ provided for every monic polynomial $p(t)$ of degree n, with coefficients from $\mathbb{F}$, there exists a matrix with entries from $\mathbb{F}$, in the given pattern, that has characteristic polynomial $p(t)$. Let $\mathbb{E} \subseteq \mathbb{F} \subseteq \mathbb{K}$ be an extension of fields. It is natural to ask whether a pattern that is spectrally arbitrary over $\mathbb{F}$ must also be spectrally arbitrary over $\mathbb{E}$ or $\mathbb{K}$. In this article it is shown that if $\mathbb{F}$ is dense in $\mathbb{K}$ and $\mathbb{K}$ is a complete metric space, then any spectrally arbitrary or relaxed spectrally arbitrary pattern over $\mathbb{F}$ is relaxed spectrally arbitrary over $\mathbb{K}$. It is also established that if $\mathbb{E}$ is an algebraically closed subfield of a field $\mathbb{F}$, then any spectrally arbitrary pattern over $\mathbb{F}$ is spectrally arbitrary over $\mathbb{E}$. The $2n$ Conjecture and the Superpattern Conjecture are explored over fields other than the real numbers. In particular, examples are provided to show that the Superpattern Conjecture is false over the field with 3 elements.

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1. Introduction

A matrix pattern $\mathscr{A}$ is a square matrix whose entries are elements that come from one of the sets $\{0, +, -\}$ (sign patterns), $\{0, *\}$ (zero–nonzero patterns), or $\{0, \sharp\}$ (relaxed zero–nonzero patterns), where $*$ denotes a nonzero entry and $\sharp$ denotes an entry that is either zero or nonzero. The idea of a spectrally arbitrary sign pattern was first introduced by Drew, Johnson, Olesky, and van den Driessche [7] in 2000. Since 2000 many properties of these patterns have been analyzed and techniques have been developed to determine whether a sign pattern is spectrally arbitrary over $\mathbb{R}$ (see for example [2,3,5,11]). The first paper on zero–nonzero spectrally arbitrary patterns was published in 2005 by Corpuz and McDonald [6]. McDonald and Yielding [12] analyzed zero–nonzero patterns over $\mathbb{C}$ and Bodine, McDonald [1], Shader [14] and others have worked on classifying spectrally arbitrary zero–nonzero patterns over finite fields. In 2003, Elsner and Hershkowitz [9] considered a particular relaxed spectrally arbitrary sign pattern, with Cavers and Fallat [4] later providing a more thorough analysis of properties gained or lost by relaxing the nonzero requirement on the starred or signed entries of a pattern.

Suppose $\mathbb{E} \subseteq \mathbb{F} \subseteq \mathbb{K}$ is an extension of fields and a pattern $\mathscr{A}$ is spectrally arbitrary over $\mathbb{F}$. It is natural to ask if $\mathscr{A}$ is also spectrally arbitrary over $\mathbb{E}$ or $\mathbb{K}$.

Of particular interest are the relationships between spectrally arbitrary patterns over $\mathbb{Q}$ (the field of rational numbers), $\mathbb{A}$ (the algebraic number field), $\mathbb{R}$ (the real numbers), and $\mathbb{C}$ (the complex numbers). In Example 14, a pattern that is spectrally arbitrary over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{A}$, but not $\mathbb{Q}$, is presented, along with a pattern that is spectrally arbitrary over $\mathbb{A}$ and $\mathbb{C}$, but not $\mathbb{Q}$ or $\mathbb{R}$. Recall that the fields $\mathbb{R}$ and $\mathbb{C}$ are complete metric spaces, whereas $\mathbb{Q}$ and $\mathbb{A}$ are not. The fields $\mathbb{A}$ and $\mathbb{C}$ are algebraically closed, whereas $\mathbb{Q}$ and $\mathbb{R}$ are not. It is also interesting to consider the fact that both $\mathbb{R}$ and $\mathbb{C}$ contain transcendental numbers, whereas $\mathbb{Q}$ and $\mathbb{A}$ do not. If no transcendental numbers are used as coefficients will transcendental numbers ever be necessary to solve a system of multivariate polynomials with algebraic coefficients? This question directly pertains to zero–nonzero matrix patterns over $\mathbb{Q}$, as each component of the polynomial map of a matrix pattern (see Definition 3) is a multivariate polynomial with rational coefficients. It is true that transcendental numbers can solve such a system (consider $f = x + y$ and the solutions $(\pi, -\pi)$ and $(1, -1)$), but are they necessary? We prove in section 3 that transcendental numbers are, in fact, not needed to solve such systems of polynomial equations.

Working in the context of fields in general, we establish in section 3 that if a field $\mathbb{F}$ is dense in a complete metric space $\mathbb{K}$, then every spectrally arbitrary or relaxed spectrally arbitrary patten over $\mathbb{F}$ is relaxed spectrally arbitrary over $\mathbb{K}$. We also establish that if $\mathbb{E}$ is algebraically closed subfield of $\mathbb{F}$, then every spectrally arbitrary pattern over $\mathbb{F}$ is spectrally arbitrary over $\mathbb{E}$.

The problem of determining the lower bound on the number of nonzero entries in a spectrally arbitrary pattern was first proposed in [2], where it is shown that the minimum number of nonzero entries in a $n \times n$ spectrally arbitrary pattern over $\mathbb{R}$ is at least $2n - 1$, but conjectured that the actual lower bound is $2n$.