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A note on the positive semidefiniteness of $A_\alpha(G)$

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ABSTRACT

Let G be a graph with adjacency matrix $A(G)$ and let $D(G)$ be the diagonal matrix of the degrees of G. For every real $\alpha \in [0,1]$, write $A_\alpha(G)$ for the matrix

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G).$$

Let $\alpha_0(G)$ be the smallest α for which $A_\alpha(G)$ is positive semidefinite. It is known that $\alpha_0(G) \leq 1/2$. The main results of this paper are:

(1) if G is d-regular then

$$\alpha_0 = \frac{-\lambda_{\min}(A(G))}{d - \lambda_{\min}(A(G))},$$

where $\lambda_{\min}(A(G))$ is the smallest eigenvalue of $A(G)$;
(2) G contains a bipartite component if and only if $\alpha_0(G) = 1/2$;
(3) if G is r-colorable, then $\alpha_0(G) \geq 1/r$.

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1. Introduction

Let G be a graph with adjacency matrix $A(G)$, and let $D(G)$ be the diagonal matrix of the degrees of G. In [8], it was proposed to study the family of matrices $A_\alpha(G)$ defined for any real $\alpha \in [0,1]$ as

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G).$$

Since $A_0(G) = A(G)$ and $2A_{1/2}(G) = Q(G)$, where $Q(G)$ is the signless Laplacian of G, the matrices A_α can help to study subtle relations between $A(G)$ and $Q(G)$.

A major distinction between $Q(G)$ and $A(G)$ is the fact that $Q(G)$ is positive semidefinite, whereas $A(G)$ is not, except if G is empty. Thus, given G, it is natural to ask for which $\alpha \in [0,1]$ is $A_\alpha(G)$ positive semidefinite. To further discuss this question we need the following notation: given a square matrix M with real eigenvalues, we write $\lambda(M)$ and $\lambda_{\min}(M)$ for the largest and the smallest eigenvalues of M.

In [8], some general results on the matrices $A_\alpha(G)$ have been proved. In particular, if $1 \geq \alpha > \beta \geq 0$, observe that

$$A_\alpha(G) - A_\beta(G) = (\alpha - \beta) L(G),$$

where $L(G) = D(G) - A(G)$ is the Laplacian of G. Hence, using Weyl's inequalities (see Theorem W below), one can get the following propositions:

Proposition 1. *Let $1 \geq \alpha > \beta \geq 0$. If G is a graph of order n with $A_\alpha(G) = A_\alpha$, $A_\beta(G) = A_\beta$, and $L(G) = L$, then*

$$(\alpha - \beta)\lambda(L) > \lambda_{\min}(A_\alpha) - \lambda_{\min}(A_\beta) \geq 0. \quad (1)$$

If G is connected, then the right inequality in (1) is strict.

Proposition 2. *If $\alpha \geq 1/2$, then $A_\alpha(G)$ is positive semidefinite. If $\alpha > 1/2$ and G has no isolated vertices, then $A_\alpha(G)$ is positive definite.*

In particular, inequalities (1) imply that for every G, the function $f_G(\alpha) := \lambda_{\min}(A_\alpha(G))$ is continuous and nondecreasing in α. Thus, there is a smallest value $\alpha \in [0,1]$ such that $\lambda_{\min}(A_\alpha(G)) = 0$. Denote this value by $\alpha_0(G)$ and note that $A_\alpha(G)$ is positive semidefinite if and only if $\alpha_0(G) \leq \alpha \leq 1$.

Having $\alpha_0(G)$ in hand, we restate a problem that has been raised in [8]:

Problem 3. Given a graph G, find $\alpha_0(G)$.

This problem seems difficult in general, but is worth studying, because $\alpha_0(G)$ relates to various structural parameters of G. In this paper, we find $\alpha_0(G)$ if G is regular or

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