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Frame vector multipliers for finite group representations[☆]

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A B S T R A C T

A frame vector (or generator) for a group representation π of a countable or finite group G on a Hilbert space H is a vector $\xi \in H$ such that $\{\pi(g)\xi\}_{g\in G}$ is a Parseval frame for H. Frame vector multipliers are the unitary operators on H that map frame vectors to frame vectors. Based on a characterization of frame vectors with respect to the standard decomposition of a group representation as the direct sums of irreducible subrepresentations (with multiplicity), we obtain explicit characterizations of frame generator multipliers for two basic cases for finite group representations. With the help of these characterizations we obtain some necessary conditions of frame vector multipliers for general frame representations, and present several examples to demonstrate how these results can be used to get explicit characterizations for frame vector multipliers.

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1. Introduction

The theory of frames has become an active research topic in the last twenty years due to its important applications in engineering (e.g. [2–5]), its connections with the theory of wavelet and time-frequency analysis and its connections with some longstanding problems, for example the recently settled Kadison–Singer problem (cf. [9]), in both pure and applied mathematics. A large amount of research on frame theory has been naturally focused on frames with special structures, not only because these frames are usually more suitable for applications but also because they have intrinsic connections with the representation theory and operator algebras (cf. [15–17,24,28]). Typical examples of frames with special structures include wavelet frames, Gabor frames and group representation frames (cf. [10,11,20,21]). All these frames are generated by applying a collection (system) of unitary operators to a single vector (or several vectors in the multi-frame case). Such vectors are called frame vectors, or wandering vectors in the case that the generated set of vectors is an orthonormal basis. Frame vector generators play important roles in establishing the connections between the structured frame theory and the theory of operators and operator algebras. For example, the wandering vector theory for unitary systems established by Xingde Dai and David Larson in [11] has direct impact and applications in wavelet and Gabor analysis, and their functional analysis approach to wavelet theory stimulated extensive research activities on the topological and geometric properties of wavelets (cf. [8,26,36]), the research on the existence and constructions of wavelet sets (cf. [7,14]) and on group representation frames (cf. [11,24,25]).

One of the interesting problems in wavelet theory is the connectivity problem of orthonormal wavelets: *Is the set of all the orthonormal wavelets path-connected in the $L^2(\mathbb{R})$-norm*? While this problem remains open, several important progresses have been made in the last 15 years. A notable result is the connectivity of Multiresolution Analysis (MRA) wavelets which was proved in [36], and its proof relied on a characterization of Fourier wavelet multipliers. Moreover, the Fourier wavelet multipliers (a measurable function f is called a Fourier wavelet multiplier if the inverse Fourier transform of $(f\widehat{\psi})$ is a wavelet for any wavelet ψ) coincide with both MRA wavelet multipliers (i.e., functions that send MRA wavelets to MRA wavelets) and the scaling function multipliers (i.e., functions that send scaling functions to scaling functions). Fourier wavelet multipliers belong to a special class of wandering vector multipliers for the unitary system $\{D^nT^m : m, n \in \mathbb{Z}\}$, where $Df(t) = \sqrt{2}f(2t)$ and $Tf(t) = f(t-1)$ are the dilation and translation operators on $L^2(\mathbb{R})$. Another important case is the case when the unitary system is a group, or an image of a (projective) unitary group representation. This led to the investigation on the theory of group representation frames and its connections with the representation theory and the theory of operator algebras (cf. [23–25,37,38]).

The structure of wandering vector multipliers for group representations was investigated in [25,29] where it was proved that the wandering vector multipliers for a group representation form a multiplicative group. The proof of this result surprisingly involves some deep theory and techniques from C*-algebras or von Neumann algebras developed