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Local Lie derivations of factor von Neumann algebras ☆

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ABSTRACT

Let $\mathcal{A}$ be a factor von Neumann algebra acting on a complex Hilbert space H with $\dim(\mathcal{A}) \geq 2$. We show that each local Lie derivation from $\mathcal{A}$ into itself is a Lie derivation.

1. Introduction

Let $\mathcal{A}$ be an algebra over the complex field $\mathbb{C}$. A linear map $\varphi : \mathcal{A} \to \mathcal{A}$ is called a derivation if $\varphi(AB) = \varphi(A)B + A\varphi(B)$ for all $A, B \in \mathcal{A}$, and φ is called a local derivation if for each $A \in \mathcal{A}$, there exists a derivation φ_A of $\mathcal{A}$, depending on A, such that $\varphi(A) = \varphi_A(A)$. Several authors have considered the relationship between local

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derivations and derivations on self-adjoint algebras or non-self-adjoint algebras, see for example [1,3,5,8,11,15,16] and the references therein. In [10], Kadison proved that every norm-continuous local derivation from a von Neumann algebra into its dual normal bimodule is a derivation. In [12], Larson and Sourour obtained the same result for $B(X)$, the algebra of all bounded linear operators on a Banach space X. In [9], Johnson extended Kadison's result to local derivations from any C^*-algebra into its Banach bimodule. In [6], Hadwin and Li proved that any bounded local derivation from a CSL algebra into its Banach bimodule is a derivation.

A linear map φ of $\mathcal{A}$ is called a Lie derivation if $\varphi([A,B]) = [\varphi(A), B] + [A, \varphi(B)]$ for all $A, B \in \mathcal{A}$, where $[A,B] = AB - BA$ is the usual Lie product. We say that a linear map from $\mathcal{A}$ into itself is a local Lie derivation if for each $A \in \mathcal{A}$, there exists a Lie derivation φ_A of $\mathcal{A}$, depending on A, such that $\varphi(A) = \varphi_A(A)$. Chen et al. [2] studied local Lie derivations of operator algebras on Banach spaces. Noticing that the proof in [2] depends heavily on rank one operators in $B(X)$, but von Neumann algebras need not contain rank one operators. It is clear that any attempt to extend Chen's results to general operator algebras must use different techniques. The purpose of the present paper is to study local Lie derivations of factor von Neumann algebras.

Let $\mathcal{A}$ be a factor von Neumann algebra acting on a complex Hilbert space H. By factor we mean that the center of $\mathcal{A}$ is $\mathbb{C}I$, where I is the identity of $\mathcal{A}$. It follows from [4] that every operator $A \in \mathcal{A}$ can be written as finite linear combinations of projections in $\mathcal{A}$. We refer the reader to [14] for the basic theory of von Neumann algebras.

We close this section with a well known result concerning Lie derivations.

Proposition 1.1. *([13]) Let $\mathcal{A}$ be a von Neumann algebra. If $\varphi : \mathcal{A} \to \mathcal{A}$ is a Lie derivation, then $\varphi = d + \tau$, where d is an associative derivation and τ is a linear map from $\mathcal{A}$ into its center vanishing on each commutator.*

2. Main results

Our main result reads as follows.

Theorem 2.1. *Let $\mathcal{A}$ be a factor von Neumann algebra acting on a complex Hilbert space H with* $\dim(\mathcal{A}) \geq 2$. *Then each local Lie derivation φ from $\mathcal{A}$ into itself is a Lie derivation.*

If $\dim(\mathcal{A}) < \infty$, then $\mathcal{A}$ is isomorphic to $M_n(\mathbb{C})$. It follows from Theorem 2.1 of [2] that each local Lie derivation of $\mathcal{A}$ is a Lie derivation. In the following, $\mathcal{A}$ is a infinite dimensional factor von Neumann algebra. For any $A \in \mathcal{A}$, the symbol φ_A stands for a Lie derivation from $\mathcal{A}$ into itself such that $\varphi(A) = \varphi_A(A)$.

Claim 2.2. *For every idempotents $P, Q \in \mathcal{A}$ and $X \in \mathcal{A}$, there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$ such that*