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Inequalities related to partial transpose and partial trace

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In this paper, we present inequalities related to partial transpose and partial trace for positive semidefinite matrices. Some interesting results involving traces and eigenvalues are also included.

1. Introduction

Throughout the paper, we use the following standard notation:

- $\mathbb{M}_{n\times k}$ is the set of $n \times k$ complex matrices; if $n = k$, we use $\mathbb{M}_n$ for $\mathbb{M}_{n\times n}$ and if $k = 1$, we use $\mathbb{C}^n$ for $\mathbb{M}_{n\times 1}$.
- $\mathbb{M}_n(\mathbb{M}_k)$ is the set of $n \times n$ block matrices with each block in $\mathbb{M}_k$.

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- I_n is the $n \times n$ identity matrix.
- $A \circ B$ is the Hadamard product of A, B.
- $A \otimes B$ is the Kronecker product of A, B; that is, if $A = [a_{ij}] \in \mathbb{M}_n$ and $B \in \mathbb{M}_k$, then $A \otimes B \in \mathbb{M}_n(\mathbb{M}_k)$ whose (i, j) block is $a_{ij}B$.
- $||x||$ denotes the 2-norm of $x \in \mathbb{C}^n$; that is, $||x|| = \sqrt{x^*x}$.

Given $A = [A_{i,j}]_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$, we define the partial transpose of A by

$$A^\tau = [A_{j,i}]_{i,j=1}^n .$$

Note that $A \geq 0$ does not necessarily imply $A^\tau \geq 0$. If both A and A^τ are positive semidefinite, then A is said to be positive partial transpose (PPT for short). We also define two partial traces $\mathrm{tr}_1 A$ and $\mathrm{tr}_2 A$ of $A = [A_{i,j}]_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ by

$$\mathrm{tr}_1 A = \sum_{i=1}^n A_{i,i},$$
$$\mathrm{tr}_2 A = [\mathrm{tr} A_{i,j}]_{i,j=1}^n ,$$

where $\mathrm{tr} X$ denotes the trace of X; see [5] for more details related to partial traces.

It is shown in [1,4] that for any positive semidefinite $H \in \mathbb{M}_n(\mathbb{M}_k)$, we have

$$I_n \otimes \mathrm{tr}_1 H + \mathrm{tr}_2 H \otimes I_k - H \leq (\mathrm{tr} H) I_{nk}. \tag{1.1}$$

Generally, $I_n \otimes \mathrm{tr}_1 H + \mathrm{tr}_2 H \otimes I_k \geq H$ does not hold. In this paper, we show

$$I_n \otimes \mathrm{tr}_1 H + \mathrm{tr}_2(H^\tau) \otimes I_k \geq H^\tau$$

for $H \in \mathbb{M}_n(\mathbb{M}_k)$ with $H \geq 0$ (in Theorem 2) and

$$I_2 \otimes \mathrm{tr}_1 H + \mathrm{tr}_2(H) \otimes I_k \geq H$$

for $H \in \mathbb{M}_2(\mathbb{M}_k)$ with $H \geq 0$ (in Theorem 4). Moreover, some interesting inequalities involving the trace and the minimum and maximum eigenvalues of a positive semidefinite matrix will be presented.

2. Results and proofs

Lemma 1. *Let $A, B \in \mathbb{M}_n$. If $A, B \geq 0$, then $\sum(A \circ B) \geq 0$, where $\sum(X)$ denotes the sum of all entries of X.*

Proof. By the Schur product theorem, we have $A \circ B \geq 0$. Since $\sum(X) = u^* X u$, where $u \in \mathbb{C}^n$ is the vector with all entries equal to one, $\sum(A \circ B) \geq 0$ follows from $A \circ B \geq 0$. $\square$