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The number of distinct eigenvalues for which an index decreases multiplicity [☆]

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ABSTRACT

For an Hermitian matrix A whose graph is a tree T, we study the number of eigenvalues of A whose multiplicity decreases when a particular vertex is deleted from T. Explicit results are given when that number of eigenvalues is less than 4 and an inductive result thereafter. The work is based, in part, on classical results about multiplicities, but also on some new facts, including a useful identity. This allows us to give strong bounds based on simple facts about the location of the vertex in the tree. Some facts about matrices whose graphs are not trees are included, and the classical diameter bound about the number of distinct eigenvalues for a tree follows.

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Eigenvalue multiplicity
Tree

1. Introduction

Given an undirected graph G on n vertices, $\mathcal{H}(G)$ denotes all the complex Hermitian matrices with graph G; no restriction is placed upon the diagonal entries. Let $\mathrm{m}_A(\lambda)$ denote the multiplicity of λ as an eigenvalue of A. Vertex i of G is called a *Parter* (resp. *neutral*, *downer*) vertex for λ in $A \in \mathcal{H}(G)$ if $m_{A(i)}(\lambda) = m_A(\lambda) + 1$ (resp. $m_A(\lambda)$, $m_A(\lambda) - 1$). Here, as usual, $A(i)$ is the $(n-1)$-by-$(n-1)$ principal submatrix of A resulting from deletion of row and column i (vertex i from G). Because of the interlacing inequalities, these are the only 3 possibilities. Here, our purpose is to focus on a particular vertex and ask for how many eigenvalues it is a downer. In the process, some striking facts about paths and downer vertices in trees, which are the main focus, are obtained. These allow very different proofs of key classical facts in the subject. Note that, using Geršgorin's theorem, it is easy to construct $A \in \mathcal{H}(T)$, T a tree, with distinct eigenvalues and all vertices downers. But we are interested in lower bounds.

See [2] for basic matrix theoretic background assumed herein. See also references [5–7] for examples of how the Parter–Wiener, etc. technology has been used in ways we employ it here. See [3,8] for basic theory about maximum multiplicity and the minimum number of distinct eigenvalues that underlay our discussion.

Throughout, T will denote a tree, and when we delete vertex i, we obtain the induced subgraph $T(i)$. By $\mathrm{d}(T)$ we mean the *diameter* of T, measured in vertices on a path that attains the diameter. As usual, we also refer to such a path as a diameter. When we wish to discuss matrices in $\mathcal{H}(T)$, we may do so by describing a particular assignment of some eigenvalues to paths or vertices of T. We often move freely between matrices and graphs, for convenience and without danger of confusion.

We denote by $\mathrm{c}(A)$ the number of distinct eigenvalues of $A \in \mathcal{H}(G)$ and by $\mathrm{c}(G) = \min_{A \in \mathcal{H}(G)} \mathrm{c}(A)$, the minimum number of distinct eigenvalues among matrices in $\mathcal{H}(G)$. It is well known that

$$\mathrm{c}(T) \geq \mathrm{d}(T)$$

for any tree T [8]. Equality occurs often, but not always. From our results a new and simple proof of this fact follows, in Section 5.

In the next section, we set some further notation that we need, and then give some key preliminary facts that we will use, including a characterization of the case in which a vertex is a downer for just two eigenvalues. In Section 3, we describe the situations in which a vertex is a downer for 3 eigenvalues and in Section 4 we give an inductive general characterization. Then in Section 5, we give bounds for how frequently a vertex is a downer, based on some simple concepts about a tree.