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Coherence invariant maps on tensor products ☆

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ABSTRACT

In 1940s, Hua established the fundamental theorem of geometry of rectangular matrices which describes the general form of coherence invariant bijective maps on the space of all matrices of a given size. In 1955, Jacob generalized Hua's theorem to that on tensors of order two over division rings with more than two elements. We generalize Jacob's result to that on tensors of any finite order over arbitrary fields and use the result to derive a special case of a theorem on linear decomposable tensor preservers of Westwick in 1967.

1. Introduction

The main objective in this paper is an improvement of a theorem on tensors of order two to tensors of any finite order proved by Henry G. Jacob, Jr. In nineteen forties, L.K. Hua proved the fundamental theorem of geometry of rectangular matrices which describes the general form of bijective maps on the space of all matrices of a given size

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over a division ring with at least three elements that preserve coherence (also called adjacency) in both directions [1]. Later, Z. Wan and Y. Wang complemented the Hua's theorem by proving the result over the division ring with two elements [2]. See also [3,4, and references therein]. Recently, the fundamental theorem has been refined in several directions, i.e., replacing the coherence preservation in two directions to that in one direction, relaxing the bijectivity assumption, or allowing the maps act between spaces of matrices of different sizes [5–8]. Some authors have also extended Hua's theorem over finite fields in terms of graph endomorphisms such as the one found in [9]. By contrast, Jacob's result generalized Hua's theorem in the other direction: from that on matrices to that on tensors by describing coherence invariant bijective maps of tensors of order two over division rings with three or more elements [10].

Throughout this paper, let $\mathbb{F}$ be a field and let $\{U_1, \ldots, U_r\}$ be a finite collection of vector spaces over $\mathbb{F}$, where $r \geq 2$. Elements of the tensor product $\bigotimes_{i=1}^r U_i$ over $\mathbb{F}$ are called tensors of *order* r. Elements of $\bigotimes_{i=1}^r U_i$ expressible in the form $\otimes_{i=1}^r u_i$ for some $(u_1, \ldots, u_r) \in \prod_{i=1}^r U_i$, the totality of which being denoted $\mathcal{D}(\bigotimes_{i=1}^r U_i)$, are called *decomposable* tensors. The zero element of $\bigotimes_{i=1}^r U_i$ is denoted 0. Elements of $\mathcal{D}(\bigotimes_{i=1}^r U_i) \backslash \{0\}$ are called *rank-one*. Every nonzero tensor is the sum of some rank-one tensors. The *rank* of a tensor A, denoted $\operatorname{rank}(A)$, is the minimum number of rank-one tensors required to yield A as their sum; $\operatorname{rank}(0) = 0$ by definition. Two tensors A, B are called *coherent* provided $A - B$ is rank-one. If a bijective map and its inverse of tensors carry pairs of coherent tensors to pairs of coherent tensors, the map is called *coherence invariant (in two directions)*.

An obvious class of coherence invariant maps are induced by permuting U_i's in the tensor product $\bigotimes_{i=1}^r U_i$. Let π be a permutation of $\{1, \ldots, r\}$. In the context of the tensor product $\bigotimes_{i=1}^r U_i$, permuting U_i's using π means replacing each decomposable tensor $\otimes_{i=1}^r u_i$ in $\bigotimes_{i=1}^r U_i$ with its counterpart $\otimes_{i=1}^r u_{\pi(i)}$ in $\bigotimes_{i=1}^r U_{\pi(i)}$. Such a process of permutation induces a coherence invariant linear isomorphism between $\bigotimes_{i=1}^r U_i$ and $\bigotimes_{i=1}^r U_{\pi(i)}$.

Example 1.1. Let π be a permutation of $\{1, \ldots, r\}$. Then there is a unique linear isomorphism $\pi_* : \bigotimes_{i=1}^r U_i \to \bigotimes_{i=1}^r U_{\pi(i)}$ such that $\pi_*(\otimes_{i=1}^r u_i) = \otimes_{i=1}^r u_{\pi(i)}$ for every $\otimes_{i=1}^r u_i \in \mathcal{D}(\bigotimes_{i=1}^r U_i)$. Furthermore, $\operatorname{rank}(\pi_*(A)) = \operatorname{rank}(A)$ for all $A \in \bigotimes_{i=1}^r U_i$ and whence π_* is coherence invariant.

To construct π_*, consider the multilinear map $\pi_\# : \prod_{i=1}^r U_i \to \bigotimes_{i=1}^r U_{\pi(i)}$ such that $\pi_\#(u_1, \ldots, u_r) = \otimes_{i=1}^r u_{\pi(i)}$ for all $(u_1, \ldots, u_r) \in \prod_{i=1}^r U_i$. By the universal property of tensor product, $\pi_\#$ induces a unique linear map π_* such that $\pi_*(\otimes_{i=1}^r u_i) = \otimes_{i=1}^r u_{\pi(i)}$ for every $\otimes_{i=1}^r u_i \in \mathcal{D}(\bigotimes_{i=1}^r U_i)$. It is readily checked that $\pi_*^{-1} = \left(\pi^{-1}\right)_*$. Hence π_* is an isomorphism.

Let $A = \sum_{j=1}^k \otimes_{i=1}^r u_{i,j} \in \bigotimes_{i=1}^r U_i$, where $k = \operatorname{rank}(A)$. Then

$$\operatorname{rank}(\pi_*(A)) = \operatorname{rank}\left(\sum_{j=1}^k \otimes_{i=1}^r u_{\pi(i),j}\right) \leq k = \operatorname{rank}(A).$$