Linear Algebra and its Applications 516 (2017) 69–81

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On domains of noncommutative rational functions

Jurij Volčič [1]

Department of Mathematics, University of Auckland, New Zealand

ARTICLE INFO

Article history:
Received 15 August 2016
Accepted 19 November 2016
Available online 23 November 2016
Submitted by P. Semrl

MSC:
primary 16S38, 93B55
secondary 47A56, 93B20, 15A54

Keywords:
Noncommutative rational function
Extended domain
Singularities
Minimal realization
Free function theory

ABSTRACT

In this paper the stable extended domain of a noncommutative rational function is introduced and it is shown that it can be completely described by a monic linear pencil from the minimal realization of the function. This result amends the singularities theorem of Kalyuzhnyi-Verbovetskyi and Vinnikov. Furthermore, for noncommutative rational functions which are regular at a scalar point it is proved that their domains and stable extended domains coincide.

1. Introduction

Noncommutative rational fractions are the elements of the universal skew field of a free algebra [9]. While this skew field can be constructed in various ways [2,20,21], it can also be defined through evaluations of formal rational expressions on tuples of matrices [16]. This interpretation gives rise to prominent applications of noncommutative rational functions in free analysis [17,1], free real algebraic geometry [14,23,7,12] and control theory [3,6]. The consideration of matrix evaluations naturally leads to the intro-

E-mail address: jurij.volcic@auckland.ac.nz.

[1] Supported by University of Auckland Doctoral Scholarship.

http://dx.doi.org/10.1016/j.laa.2016.11.031

duction of the *domain* of a noncommutative rational function. However, at first sight this notion seems intangible: since a noncommutative rational function is an equivalence class of formal rational expressions, its domain is defined as the union of the formal domains of all its representatives; see Subsection 2.1 for precise definition. Therefore new variants of domains emerged: *extended domains* [15,16] and *analytic* or *limit domains* [14,12]. Both of these notions are related to generic evaluations of noncommutative rational functions and can thus be described using a single representative of a function.

The main important breakthrough in characterizing domains was done by Dmitry Kaliuzhnyi-Verbovetskyi and Victor Vinnikov in [15]. Perceptively combining linear systems realizations from control theory and difference-differential operators from free analysis they seemingly proved that the extended domain of a noncommutative rational function $\mathbb{r}$ that is regular at the origin coincides with the invertibility set of the monic linear pencil from a minimal realization of $\mathbb{r}$; see [15, Theorem 3.1]. While minimal realizations of noncommutative rational functions can be effectively computed [3,8], monic linear pencils are key tools in matrix theory and are well-explored through control theory [4], algebraic geometry [10] and optimization [25]. Therefore the result of Kaliuzhnyi-Verbovetskyi and Vinnikov proved to be of great importance in free real algebraic geometry and free function theory [13,6,22,12,18,19]. Alas, there is a gap in its proof and its conclusion does not hold. The reason behind is the fact that the extended domain of a noncommutative rational function is in general not closed under direct sums; a concrete instance when this occurs is given in Example 2.1.

The main results of this paper adjust and improve [15, Theorem 3.1]. First we recall domains and extended domains in Subsection 2.1 and give the necessary facts about realizations in Subsection 2.2. Then we define the *stable extended domain* of a noncommutative rational function (Definition 3.1). In Proposition 3.3 it is shown that the stable extended domain is always closed under direct sums, which is in contrast with the extended domain. In Theorem 3.5 we prove that the following variant of [15, Theorem 3.1] holds.

If a noncommutative rational function $\mathbb{r}$ is regular at the origin, then its stable extended domain is equal to the invertibility set of the monic linear pencil from the minimal realization of $\mathbb{r}$.

This statement is then extended to noncommutative rational functions that are regular at some scalar point in Corollary 3.7. Moreover, for such functions we are able to completely describe their domains due to the following result.

If a noncommutative rational function is regular at some scalar point, then its domain and its stable extended domain coincide.

See Theorem 3.10 for the proof. Finally, in Example 3.13 we present a noncommutative rational function whose domain is strictly larger than the domain of any of its representatives.