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Decay bounds for the numerical quasiseparable preservation in matrix functions ☆

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ABSTRACT

Given matrices A and B such that $B = f(A)$, where $f(z)$ is a holomorphic function, we analyze the relation between the singular values of the off-diagonal submatrices of A and B. We provide a family of bounds which depend on the interplay between the spectrum of the argument A and the singularities of the function. In particular, these bounds guarantee the numerical preservation of quasiseparable structures under mild hypotheses. We extend the Dunford–Cauchy integral formula to the case in which some poles are contained inside the contour of integration. We use this tool together with the technology of hierarchical matrices ($\mathcal{H}$-matrices) for the effective computation of matrix functions with quasiseparable arguments.

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1. Introduction

Matrix functions are an evergreen topic in matrix algebra due to their wide use in applications [17,20,24,26,27]. It is not hard to imagine why the interaction of structures with matrix functions is an intriguing subject. In fact, in many cases structured matrices arise and can be exploited for speeding up algorithms, reducing storage costs or allowing to execute otherwise not feasible computations. The property we are interested in is the *quasi-separability*. That is, we want to understand whether the submatrices of $f(A)$ contained in the strict upper triangular part or in the strict lower triangular part, called *off-diagonal submatrices*, have a "small" numerical rank.

Studies concerning the numerical preservation of data-sparsity patterns were carried out recently [1–3,11]. Regarding the quasiseparable structure [14,15,32,33], in [18,19,22] Gavrilyuk, Hackbusch and Khoromskij addressed the issue of approximating some matrix functions using the hierarchical format [9]. In these works the authors prove that, given a low rank quasiseparable matrix A and a holomorphic function $f(z)$, computing $f(A)$ via a quadrature formula applied to the contour integral definition, yields an approximation of the result with a low quasiseparable rank. Representing A with a $\mathcal{H}$-matrix and exploiting the structure in the arithmetic operations provides an algorithm with almost linear complexity. The feasibility of this approach is equivalent to the existence of a rational function $r(z) = \frac{p(z)}{q(z)}$ which well-approximates the holomorphic function $f(z)$ on the spectrum of the argument A. More precisely, since the quasiseparable rank is invariant under inversion and sub-additive with respect to matrix addition and multiplication, if $r(z)$ is a good approximation of $f(z)$ of low degree then the matrix $r(A)$ is an accurate approximation of $f(A)$ with low quasiseparable rank. This argument explains the preservation of the quasiseparable structure, but still needs a deeper analysis which involves the specific properties of the function $f(z)$ in order to provide effective bounds to the quasiseparable rank of the matrix $f(A)$.

In this article we deal with the analysis of the quasiseparable structure of matrix functions by studying the interplay between the off-diagonal singular values of the matrices A and B such that $B = f(A)$. Our intent is to understand which parameters of the model come into play in the numerical preservation of the structure and to extend the analysis to functions with singularities.

In Section 2 we see how the integral definition of a matrix function enables us to study the structure of the off-diagonal blocks in $f(A)$. In Section 3 we develop the analysis of the singular values of structured outer products and we derive bounds for the off-diagonal singular values of matrix functions.

In Section 4 we adapt the approach to treat functions with singularities.

The key role is played by an extension of the Dunford–Cauchy formula to the case in which some singularities lie inside the contour of integration. In Section 5 we comment on computational aspects and we perform some experiments for validating the theoretical results, while in Section 6 we give some concluding remarks.