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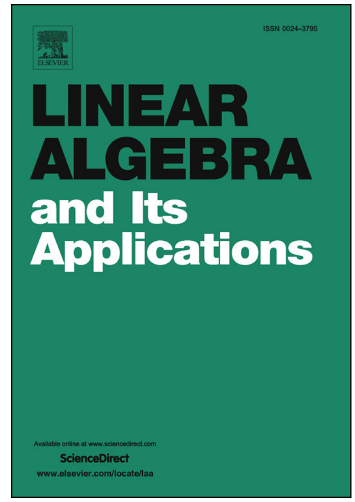

Matrices similar to partial isometries

Stephan Ramon Garcia, David Sherman

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MATRICES SIMILAR TO PARTIAL ISOMETRIES

STEPHAN RAMON GARCIA AND DAVID SHERMAN

Abstract. We determine when a matrix is similar to a partial isometry, refining a result of Halmos–McLaughlin.

1. Introduction

A Hilbert space operator V is a *partial isometry* if the restriction of V to $(\ker V)^{\perp}$ is isometric. For a complex matrix, this means that all of its singular values are in $\{0,1\}$, or in other words, the positive semidefinite factor in its polar decomposition is an orthogonal projection. These properties are not preserved by similarity; for example

$$\begin{bmatrix} 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{1}$$

are similar, since both matrices have the same Jordan canonical form. The first is a partial isometry since its nonzero columns are orthonormal, but the second is not. Which matrices are similar to a partial isometry?

The basic features of partial isometries were laid out over fifty years ago [2,6,7], and the similarity question is not new – but most work has focused on the (still unresolved) infinite-dimensional case, e.g., [3]. In the finite-dimensional case, the best result was a theorem of Halmos–McLaughlin stating that the characteristic polynomial of a nonunitary partial isometry can be any monic polynomial whose roots lie in the closed unit disk and include zero [6, Theorem 3]. The referee pointed out that this can be deduced directly from the Weyl–Horn inequalities [8,13], which say that there exists an $n \times n$ matrix with prescribed singular values $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n \geq 0$ and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, indexed so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, if and only if

$$\sigma_1\sigma_2\cdots\sigma_k \geq |\lambda_1\lambda_2\cdots\lambda_k| \qquad \text{for } k = 1,2,\ldots,n-1$$

and

$$\sigma_1\sigma_2\cdots\sigma_n = |\lambda_1\lambda_2\cdots\lambda_n|.$$

For an $n \times n$ partial isometry of rank $r < n$, we have

$$\sigma_1 = \sigma_2 = \cdots = \sigma_r = 1 \qquad \text{and} \qquad \sigma_{r+1} = \cdots = \sigma_n = 0.$$

Hence any n points (with possible repetition) in the closed unit disk can be its eigenvalues, so long as at least $n-r$ of them are zero.

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