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On the minimum trace norm/energy of $(0,1)$-matrices

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ABSTRACT

The trace norm of a matrix is the sum of its singular values. This paper presents results on the minimum trace norm $\psi_n(m)$ of $(0,1)$-matrices of size $n \times n$ with exactly m ones. It is shown that:

(1) if $n \geq 2$ and $n < m \leq 2n$, then $\psi_n(m) \leq \sqrt{m + \sqrt{2(m-1)}}$, with equality if and only if m is a prime;

(2) if $n \geq 4$ and $2n < m \leq 3n$, then $\psi_n(m) \leq \sqrt{m + 2\sqrt{2\lfloor m/3 \rfloor}}$, with equality if and only if m is a prime or a double of a prime;

(3) if $3n < m \leq 4n$, then $\psi_n(m) \leq \sqrt{m + 2\sqrt{m-2}}$, with equality if and only if there is an integer $k \geq 1$ such that $m = 12k \pm 2$ and $4k \pm 1, 6k \pm 1, 12k \pm 1$ are primes.

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1. Introduction

The *trace norm* $\|A\|_*$ of a matrix A is the sum of its singular values, also known as the *nuclear norm* or the *Schatten* 1*-norm of* A. The trace norm is an important parameter that has been intensively studied both in pure and applied mathematics.[1]

In particular, the trace norm of the adjacency matrices of graphs has been long investigated under the name *graph energy,* introduced by Gutman in [3]; for an overview of this vast research the reader is referred to [6].

In [8] it has been shown that several properties of graph energy extend to the trace norm of complex matrices, so sometimes it is compelling to call $\|A\|_*$ the *energy* of A.

A number of extremal problems about the trace norm of matrices have been presented in the survey [9], including many upper bounds on $\|A\|_*$. Since lower bounds on $\|A\|_*$ have not been studied in comparative detail, in this paper we initiate the study of the minimum trace norm of square $(0,1)$-matrices with given number of ones. This topic turns out to be both fascinating and hard; in particular, we show that some rather simple questions are tantamount to unsolved problems about prime numbers.

Thus, let the integers n and m satisfy $n \geq 2$ and $1 \leq m \leq n^2$, write $\mathbb{Z}_n(m)$ for the set of $(0,1)$-matrices of size $n \times n$ with exactly m ones, and set

$$\psi_n(m) = \min \{\|A\|_* : A \in \mathbb{Z}_n(m)\}.$$

We are interested in the following natural problem:

Problem 1. Find $\psi_n(m)$ for all admissible n and m.

It is not hard to see that $\psi_n(m) \geq \sqrt{m}$; in fact, writing $|A|_2$ for the Frobenius norm of a matrix A, one can come up with the following simple result (see, e.g., Theorem 4.3 of [9]):

If A is a complex matrix, then $\|A\|_* \geq |A|_2$. Equality holds if and only if the rank of A is 1.

It is trivial to construct a complex matrix of rank 1 with arbitrary Frobenius norm $|A|_2$, but this is not always possible should the matrix belong to $\mathbb{Z}_n(m)$, e.g., $\mathbb{Z}_3(5)$ contains no matrix of rank 1. Consequently, finding $\psi_n(m)$ turns out to be a subtle and challenging problem, sometimes leading to extremely difficult number-theoretical questions. We solved Problem 1 for $m \leq 3n$ and partially solved it for $3n < m \leq 4n$; even at that early stage it becomes clear that the full solution of Problem 1 is beyond the reach of present day mathematics.

Before stating our main results, note that the case $1 \leq m \leq n$ is trivial, as any $n \times n$ matrix with m ones in a single row or column implies that $\psi_n(m) = \sqrt{m}$. In contrast, the cases $n < m \leq 2n$ and $2n < m \leq 3n$ are far from obvious.

[1] The trace norm has been introduced for abstract operators by Schatten [10] and Grothendieck [2], and has been used in functional analysis ever since then. On the other hand, see [5] and [7] for two of the numerous applied studies of the trace norm.