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Maximal rank in matrix spaces via graph matchings

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Let $M_n(\mathbb{F})$ be the space of $n \times n$ matrices over a field $\mathbb{F}$. A matrix $A = \big(A(i,j)\big)_{i,j=1}^n \in M_n(\mathbb{F})$ is weakly symmetric if $A(i,j) \neq 0$ iff $A(j,i) \neq 0$ holds for all i, j. A matrix is alternating if it is skew-symmetric with zero diagonal. Let $W_n(\mathbb{F})$ and $A_n(\mathbb{F})$ denote respectively the set of weakly symmetric matrices and the space of alternating matrices in $M_n(\mathbb{F})$. Let $[n] = \{1, \ldots, n\}$. For $0 \neq A \in W_n(\mathbb{F})$ let $\tilde{q}(A) = \{i, j\}$, where (i, j) is the unique pair in $[n]^2$ such that $A(i,j) \neq 0$ and $A(i',j') = 0$ whenever $j < j'$ or $j = j'$ and $i < i'$. For a translate $\mathcal{S}$ of a linear space $\mathcal{B} \subset W_n(\mathbb{F})$ let $G_{\mathcal{S}}$ be the graph with loops on the vertex set $[n]$ with edge set $E_{\mathcal{S}} = \{\tilde{q}(B) : 0 \neq B \in \mathcal{B}\}$. A subset $M \subset E_{\mathcal{S}}$ is a matching if $e \cap e' = \emptyset$ for all $e \neq e' \in M$. Let $\mu(G_{\mathcal{S}}) = \max \sum_{e \in M} |e|$ where M ranges over all matchings $M \subset E_{\mathcal{S}}$. Let $\rho(\mathcal{S})$ denote the maximal rank of a matrix in $\mathcal{S}$.

It is shown that if $\mathcal{S}$ is a translate of a linear space contained in $W_n(\mathbb{F})$ and $|\mathbb{F}| \geq 3$ then $\rho(\mathcal{S}) \geq \mu(G_{\mathcal{S}})$. The restriction on $\mathbb{F}$ can be removed if $\mathcal{S}$ is an affine subspace of $A_n(\mathbb{F})$. Applications include simple proofs of upper bounds on the dimension of affine subspaces of symmetric and alternating matrices of bounded rank.

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1. Introduction

Let $M_n(\mathbb{F})$ be the space of $n \times n$ matrices over a field $\mathbb{F}$. For a subset $\emptyset \neq \mathcal{S} \subset M_n(\mathbb{F})$ let $\rho(\mathcal{S}) = \max\{\text{rank}(A) : A \in \mathcal{S}\}$ denote the maximum rank of a matrix in $\mathcal{S}$. Let $H_n(\mathbb{F})$ denote the space of symmetric matrices in $M_n(\mathbb{F})$. A matrix $A = \big(A(i,j)\big)_{i,j=1}^n \in M_n(\mathbb{F})$ is *alternating* if $A = -A^T$ and $A(i,i) = 0$ for $1 \leq i \leq n$. Let $A_n(\mathbb{F})$ denote the space of alternating matrices in $M_n(\mathbb{F})$. A matrix $A \in M_n(\mathbb{F})$ is *weakly symmetric* if $A(i,j) \neq 0$ iff $A(j,i) \neq 0$ holds for all i, j. Let $W_n(\mathbb{F})$ denote the set of all weakly symmetric matrices in $M_n(\mathbb{F})$. Note that $A_n(\mathbb{F}), H_n(\mathbb{F}) \subset W_n(\mathbb{F})$ and $W_n(\mathbb{F}_2) = H_n(\mathbb{F}_2)$. In this note we study lower bounds on $\rho(\mathcal{S})$ for affine translates $\mathcal{S}$ of linear spaces of weakly symmetric matrices, in terms of matching numbers of a certain graph associated with $\mathcal{S}$.

Let $[n] = \{1, \ldots, n\}$ and let $[n]^2_{\leq} = \{(i,j) \in [n]^2 : i \leq j\}$. The *colexicographic order* on $[n]^2_{\leq}$ is given by $(i,j) \prec (i',j')$ iff $j < j'$ or $j = j'$ and $i < i'$. Equivalently, $(i,j) \prec (i',j')$ iff $2^i + 2^j < 2^{i'} + 2^{j'}$. Let $K_n = \{e \subset [n] : |e| = 2\}$ denote the edge set of the complete graph on $[n]$ and let $\tilde{K}_n = \{e \subset [n] : 0 < |e| \leq 2\}$ denote the edge set of the complete graph with loops on $[n]$. A subset $M \subset \tilde{K}_n$ is a *matching* if $e \cap e' = \emptyset$ for all $e \neq e' \in M$. For a graph with loops $G \subset \tilde{K}_n$ let $\mathcal{M}(G)$ denote the set of all matchings $M \subset G$. Let $\nu(G) = \max\{|M| : M \in \mathcal{M}(G)\}$ and $\mu(G) = \max\{\sum_{e \in M} |e| : M \in \mathcal{M}(G)\}$. A matching $M \subset \tilde{K}_n$ is *perfect* if $\mu(M) = n$. Note that if G is loopless, i.e. $G \subset K_n$, then $\nu(G)$ is the usual matching number of G and $\mu(G) = 2\nu(G)$.

For $0 \neq A = \big(A(i,j)\big)_{i,j=1}^n \in W_n(\mathbb{F})$ let $q(A) = \max\{(i,j) \in [n]^2_{\leq} : A(i,j) \neq 0\}$ where the maximum is taken with respect to the colexicographic order. For A such that $q(A) = (i,j)$ let $\tilde{q}(A) = \{i,j\} \in \tilde{K}_n$. Let $\mathcal{S}$ be a translate of a linear space $\mathcal{B} \subset W_n(\mathbb{F})$, i.e. $\mathcal{S} = A + \mathcal{B}$ for some $A \in M_n(\mathbb{F})$. Associate with $\mathcal{S}$ a graph with loops

$$G_{\mathcal{S}} = \{\tilde{q}(B) : 0 \neq B \in \mathcal{B}\} = \{\tilde{q}(S_1 - S_2) : S_1 \neq S_2 \in \mathcal{S}\} \subset \tilde{K}_n.$$

Our main results provide a link between the maximum rank in $\mathcal{S}$ and matchings in $G_{\mathcal{S}}$.

Theorem 1.1. *Suppose $|\mathbb{F}| \geq 3$ and let $\mathcal{S} = A + \mathcal{B}$ where $A \in M_n(\mathbb{F})$ and $\mathcal{B}$ is a linear space contained in $W_n(\mathbb{F})$. Then $\rho(\mathcal{S}) \geq \mu(G_{\mathcal{S}})$.*

The restriction on $\mathbb{F}$ is superfluous if $\mathcal{S}$ is an affine space of alternating matrices:

Theorem 1.2. *Let $\mathbb{F}$ be an arbitrary field and let $\mathcal{S}$ be an affine subspace of $A_n(\mathbb{F})$. Then $\rho(\mathcal{S}) \geq \mu(G_{\mathcal{S}})$.*

Remarks. 1. The case $A = 0$ and $|\mathbb{F}| \geq \mu(G_{\mathcal{B}}) + 1$ of Theorem 1.1 had (essentially) been proved in [5]. The approach to the present improved result is somewhat different and uses additional ideas. The proof of Theorem 1.2 utilizes Pfaffians of alternating matrices.

2. Theorem 1.1 does not hold for $\mathbb{F} = \mathbb{F}_2$ as the following examples show.