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# A short note on the ratio between sign-real and sign-complex spectral radius of a real square matrix

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ABSTRACT

For a real $(n \times n)$-matrix $A$ the sign-real and the sign-complex spectral radius – invented by Rump – are respectively defined as

$$\rho^{\mathbb{R}}(A) := \max\{|\lambda| \ \mid \ |Ax| = |\lambda x|, \lambda \in \mathbb{R}, x \in \mathbb{R}^n \backslash \{0\}\},$$

$$\rho^{\mathbb{C}}(A) := \max\{|\lambda| \ \mid \ |Ax| = |\lambda x|, \lambda \in \mathbb{C}, x \in \mathbb{C}^n \backslash \{0\}\}.$$

For $n \geq 2$ we prove $\rho^{\mathbb{R}}(A) \geq \zeta_n \ \rho^{\mathbb{C}}(A)$ where the constant $\zeta_n := \sqrt{\dfrac{1-\cos(\pi/n)}{1+\cos(\pi/n)}}$ is best possible.



## 1. Introduction

Let $n \in \mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For a square matrix $A \in \mathbb{K}^{n,n}$ the so-called *generalized* spectral radius is defined by

$$\rho^{\mathbb{K}}(A) := \max\{|\lambda| \ \mid \ |Ax| = |\lambda x|, \lambda \in \mathbb{K}, x \in \mathbb{K}^n \backslash \{0\}\}.^{1} \tag{1}$$

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[1] The equality $|Ax| = |\lambda x|$ is meant componentwise.

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For $\mathbb{K} = \mathbb{R}$ it is called *sign-real* and for $\mathbb{K} = \mathbb{C}$ *sign-complex* spectral radius. The generalized spectral radius was invented and investigated by Rump [3–8].[2] The new concept was used and extended, for example, by Goldberger and Neumann [1], Peña [2], Varga [9], Zalar [10], and Zangiabadi and Afshin [11,12]. Rump gave many equivalent definitions of the generalized spectral radius, e.g. the Max–Min-characterization [8], Theorem 2.4:

$$\rho^{\mathbb{K}}(A) = \max_{x \in \mathbb{K}^n \setminus \{0\}} \min_{x_i \neq 0} \frac{|(Ax)_i|}{|x_i|} \ . \tag{2}$$

Here we are interested in bounding the ratio $\rho^{\mathbb{R}}(A)/\rho^{\mathbb{C}}(A)$ between sign-real and sign-complex spectral radius from below independent of a specific real $A$, i.e., we are searching for the maximum constant $\zeta_n \in \mathbb{R}_{\geq 0}$ that fulfills

$$\rho^{\mathbb{R}}(A) \geq \zeta_n \ \rho^{\mathbb{C}}(A) \quad \text{for all } A \in \mathbb{R}^{n,n}. \tag{3}$$

First, note that the one-dimensional case $n = 1$ is trivial since sign-real and sign-complex spectral radius of a real $(1 \times 1)$-matrix coincide. Therefore we will assume $n \geq 2$ from now on. Next, note that bounding the ratio $\rho^{\mathbb{R}}(A)/\rho^{\mathbb{C}}(A)$ from above yields 1 as best possible upper bound since by definition $\rho^{\mathbb{R}}(A) \leq \rho^{\mathbb{C}}(A)$ and $\rho^{\mathbb{R}}(A) = \rho^{\mathbb{C}}(A)$ for real diagonal matrices $A$.

In [8], Theorem 6.1, Rump proved that the skew-symmetric Toeplitz matrix

$$A := (\mathrm{sign(j-i)})_{1 \leq \mathrm{i,j} \leq \mathrm{n}} = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ -1 & \dots & -1 & 0 \end{pmatrix} \tag{4}$$

has sign-real spectral radius $\rho^{\mathbb{R}}(A) = 1$ and sign-complex spectral radius

$$\rho^{\mathbb{C}}(A) = \rho(A) = \|A\|_2 = \frac{\sin(\pi/n)}{1 - \cos(\pi/n)} = \sqrt{\frac{1 + \cos(\pi/n)}{1 - \cos(\pi/n)}} =: \zeta_n^{-1}, \tag{5}$$

where $\rho(A)$ denotes the common spectral radius, i.e., the maximum modulus of all eigenvalues. Thus $\rho^{\mathbb{R}}(A)/\rho^{\mathbb{C}}(A) = 1/\rho^{\mathbb{C}}(A) = \zeta_n$ for the matrix in (4) and $n \geq 2$.

The main goal of this short note is to prove that $\zeta_n$ as defined in (5) fulfills (3). This is demonstrated in the following main theorem:

**Theorem 1.** *For $n \in \mathbb{N}_{\geq 2}$ and every $A \in \mathbb{R}^{n,n}$ we have*

$$\rho^{\mathbb{R}}(A) \geq \zeta_n \ \rho^{\mathbb{C}}(A) \tag{6}$$

[2] The notation of the sign-real spectral radius changes from $\rho_0^S(A)$ in earlier papers to $\rho^{\mathbb{R}}(A)$.