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Subresultants in multiple roots: An extremal case

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ABSTRACT

We provide explicit formulae for the coefficients of the order-d polynomial subresultant of $(x-\alpha)^m$ and $(x-\beta)^n$ with respect to the set of Bernstein polynomials $\{(x-\alpha)^j(x-\beta)^{d-j}, 0 \leq j \leq d\}$. They are given by hypergeometric expressions arising from determinants of binomial Hankel matrices.

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1. Introduction

Let $\mathbb{K}$ be a field, and $f = f_m x^m + \cdots + f_0$ and $g = g_n x^n + \cdots + g_0$ be two polynomials in $\mathbb{K}[x]$ with $f_m \neq 0$ and $g_n \neq 0$. Set $0 \leq d < \min\{m, n\}$. The *order-d subresultant* $\operatorname{Sres}_d(f, g)$ is the polynomial in $\mathbb{K}[x]$ defined as

$$\operatorname{Sres}_d(f,g) := \det \overset{m+n-2d}{\begin{array}{|cccccc|l} \hline f_m & \cdots & & \cdots & f_{d+1-(n-d-1)} & x^{n-d-1}f & \\ & \ddots & & & \vdots & \vdots & n-d \\ & & f_m & \cdots & f_{d+1} & f & \\ \hline g_n & \cdots & & \cdots & g_{d+1-(m-d-1)} & x^{m-d-1}g & \\ & \ddots & & & \vdots & \vdots & m-d \\ & & g_n & \cdots & g_{d+1} & g & \\ \hline \end{array}}, \qquad (1)$$

where, by convention, $f_\ell = g_\ell = 0$ for $\ell < 0$.

Although it is not immediately transparent from the definition, $\operatorname{Sres}_d(f, g)$ is a polynomial of degree at most d, whose coefficients are equal to some minors of the Sylvester matrix of f and g. Subresultants were implicitly introduced by Jacobi [11] and explicitly by Sylvester [22,23], see [9] for a comprehensive historical account.[1]

For any finite subsets $A = \{\alpha_1, \ldots, \alpha_m\}$ and $B = \{\beta_1, \ldots, \beta_n\}$ of $\mathbb{K}$, and for $0 \leq p \leq m$, $0 \leq q \leq n$, one can define after Sylvester [24] the *double sum* expression:

$$\operatorname{Syl}_{p,q}(A,B)(x) := \sum_{\substack{A' \subset A,\, B' \subset B \\ |A'|=p,\, |B'|=q}} \frac{\mathcal{R}(A',B')\,\mathcal{R}(A\backslash A', B\backslash B')}{\mathcal{R}(A', A\backslash A')\,\mathcal{R}(B', B\backslash B')}\,\mathcal{R}(x,A')\,\mathcal{R}(x,B'),$$

where $\mathcal{R}(Y,Z) := \prod_{y\in Y}\prod_{z\in Z}(y-z)$.

Sylvester stated in [24], then proved in [25, Section II], the following connection between subresultants and double sums: assume that $d = p + q$, and suppose that f and g are the square-free polynomials

$$f = (x-\alpha_1)\cdots(x-\alpha_m) \quad \text{and} \quad g = (x-\beta_1)\cdots(x-\beta_n).$$

Then,

$$\binom{d}{p}\operatorname{Sres}_d(f,g) = (-1)^{p(m-d)}\operatorname{Syl}_{p,q}(f,g).$$

This identity can be regarded as a generalization to subresultants of the famous Poisson formula [18] for the resultant of f and g:

[1] The Sylvester matrix was defined in [23], and the order-d subresultant was introduced in [22,23] under the name of “prime derivative of the d-degree”.