Linear Algebra and its Applications 529 (2017) 355–373

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On Jacobian group and complexity of the generalized Petersen graph $GP(n, k)$ through Chebyshev polynomials

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A R T I C L E   I N F O

Article history:
Received 7 December 2016
Accepted 24 April 2017
Available online 29 April 2017
Submitted by R. Brualdi

MSC:
05C30
39A10

Keywords:
Spanning tree
Jacobian group
Petersen graph
Laplacian matrix
Chebyshev polynomial

A B S T R A C T

In the present paper we give a new method for calculating Jacobian group $Jac(GP(n, k))$ of the generalized Petersen graph $GP(n, k)$. We show that the minimum number of generators of $Jac(GP(n, k))$ is at least two and at most $2k + 1$. Both estimates are sharp. Also, we obtain a closed formula for the number of spanning trees of $GP(n, k)$ in terms of Chebyshev polynomials and investigate some arithmetical properties of this number.

1. Introduction

The notion of the Jacobian group of a graph, which is also known as the Picard group, the critical group, and the dollar or sandpile group, was independently introduced by

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http://dx.doi.org/10.1016/j.laa.2017.04.032

many authors ([7,3,4,2,11,14]). This notion arises as a discrete version of the Jacobian in the classical theory of Riemann surfaces. It also admits a natural interpretation in various areas of physics, coding theory, and financial mathematics. The Jacobian group is an important algebraic invariant of a finite graph. In particular, its order coincides with the number of spanning trees of the graph, which is known for some simple graphs such as the wheel, fan, prism, ladder, and Möebius ladder [5], grids [18], lattices [19], Sierpinski gaskets [6,8], 3-prism and 3-anti-prism [21]. At the same time, the structure of the Jacobian is known only in particular cases [7,4,15,24,25,16] and [17]. We mention that the number of spanning trees for circulant graphs is expressed in terms of the Chebyshev polynomials; it was found in [26,27] and [23]. We show that similar results are also true for the generalized Petersen graph $GP(n, k)$.

The generalized Petersen graph $GP(n, k)$ has vertex set and edge set given by

$$V(P(n,k)) = \{u_i, v_i \mid i = 1, 2, \ldots, n\}$$
$$E(P(n,k)) = \{u_i u_{i+1},\ u_i v_i,\ v_i v_{i+k} \mid i = 1, 2, \ldots, n\},$$

where the subscripts are expressed as integers modulo n. It is important to note that the graph $GP(n, k)$ is isomorphic to $GP(n, n-k)$, and if n and k are relative prime, then $GP(n, k)$ is also isomorphic to $GP(n, r)$ with $k\,r \equiv 1 \bmod n$ [20]. In what follows we will deal with 3-valent graphs only. So, we assume $n \geq 3$ and $1 \leq k < n/2$. The classical Petersen graph is $GP(5, 2)$. The number of spanning trees of the Petersen graph is calculated in [12] and the spectrum of the generalized Petersen graphs is obtained in [13]. Even though the number of spanning trees of a regular graph can be computed by its eigenvalues, it is not easy to obtain a closed formula for the number of spanning trees for $GP(n, k)$ by the result of [13]. In this paper, we find a closed formula for the number of spanning trees for $GP(n, k)$ through Chebyshev polynomials. Also, we suggest a new method to calculate the Jacobian group of $GP(n, k)$.

2. Basic definitions and preliminary facts

Consider a connected finite graph G, allowed to have multiple edges but without loops. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Given $u, v \in V(G)$, we set a_{uv} to be equal to the number of edges between vertices u and v. The matrix $A = A(G) = \{a_{uv}\}_{u,v\in V(G)}$ is called *the adjacency matrix* of the graph G. The degree $d(v)$ of a vertex $v \in V(G)$ is defined by $d(v) = \sum_{u\in V(G)} a_{uv}$. Let $D = D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv} = d(v)$. The matrix $L = L(G) = D(G) - A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph G.

Recall [15] the following useful relation between the structure of the Laplacian matrix and the Jacobian of a graph G. Consider the Laplacian $L(G)$ as a homomorphism $\mathbb{Z}^{|V|} \to \mathbb{Z}^{|V|}$, where $|V| = |V(G)|$ is the number of vertices in G. The cokernel $\mathrm{coker}\,(L(G)) = \mathbb{Z}^{|V|}/\mathrm{im}\,(L(G))$ is an Abelian group. Let