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On quadratic polynomial mappings of the plane

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We show that up to linear equivalence, there is only finitely many polynomial quadratic mappings $f : \mathbb{C}^2 \to \mathbb{C}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$. We list all possibilities.

1. Introduction

Let $\Omega(d_1, d_2)$ denote the space of polynomial mappings $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ where $\deg f_1 \leq d_1$ and $\deg f_2 \leq d_2$. Let $f, g \in \Omega(d_1, d_2)$. We say that f is *topologically*

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(respectively *linearly*) *equivalent* to g if there are homeomorphisms (respectively linear isomorphisms) $\Phi, \Psi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $f = \Psi \circ g \circ \Phi$. In the paper [2] it was shown that there is only a finite number of different topological types of mappings in $\Omega(d_1, d_2)$. Moreover, we know (see e.g. [10]) that there is a Zariski open dense subset $U \subset \Omega(d_1, d_2)$ such that every mapping $f \in U$ has the same, generic, topological type. If a mapping f has a generic topological type then we say that f is a *generic* mapping. In practice it is difficult to describe the generic topological type and other topological types effectively.

Here we consider the case $d_1 = d_2 = 2$. We show that in this case the topological equivalence almost coincides with the linear equivalence. Moreover, we obtain a full classification of quadratic mappings of $\mathbb{C}^2$, with respect to the linear (hence also topological) equivalence. In particular we find a model of a generic mapping of $\Omega(2, 2)$. The aim of this paper is to make a first step to understand the structure of the space $\Omega(d_1, d_2)$. General quadratic mappings independently have been intensively studied (see [1,4–7]) and this subject is interesting on its own. But to the best knowledge of the authors the classification (up to isotopy) of quadratic mappings of the plane was done only for real homogeneous mapping (see Prop. 1 in [1]) and from this point of view we fill the gap in the literature.

We explain our basic idea. It was proved in [3] that a generic polynomial mapping from $\Omega(2, 2)$ has a rational cuspidial curve of degree 4 as a discriminant, moreover this curve is tangent in two smooth points to the line at infinity. On the other hand any two such rational cuspidial curves are projectively equivalent (see [12]) hence it is easy to deduce that discriminants of generic mappings from $\Omega(2, 2)$ are linearly equivalent. Using [8] we can deduce that any two generic mappings from $\Omega(2, 2)$ are algebraically equivalent and consequently (since they have the same algebraic degree) they are linearly equivalent. Moreover, there are only finitely many possible discriminants of quadratic mappings, up to a linear equivalence (because in the non-generic case they have a non-trivial action of an infinite affine group). Hence we can expect that there is only a finite number of orbits of the action of a linear group on $\Omega(2, 2)$. We show that it is indeed the case. In fact, it can be done in an elementary way, however it is surprising that this result was not discovered up till now.

Let $C(f), \Delta(f), \mu(f)$ denote: the set of critical points, the discriminant and topological degree (see Definition 2.1). We have the following possibilities:

Generically-finite mappings:

(1) (the generic case) $f_1 = (x^2 + y, y^2 + x)$, $C(f_1) = \{4xy - 1 = 0\}$ is a hyperbola and $\Delta(f_1) = \{2^8x^2y^2 - 2^8x^3 - 2^8y^3 + 2^5 \cdot 9xy - 27 = 0\}$ is a reduced and irreducible curve with 3 cusps at points $f_1(\frac{\varepsilon}{2}, \frac{\varepsilon^2}{2})$, where $\varepsilon^3 = 1$, $\dim O(f_1) = 12$, $O(f_1)$ is an open and dense affine subvariety of $\Omega(2, 2)$, moreover $\chi(O(f_1)) = 0$ and $\mu(f_1) = 4$.

(2) $f_2 = (x^2 + y, xy)$, $C(f_2) = \{2x^2 = y\}$ is a parabola and $\Delta(f_2) = \{4x^3 = 27y^2\}$ is a cusp, $\dim O(f_2) = 11$, $O(f_2)$ is an affine subvariety of $\Omega(2, 2)$, $\mu(f_2) = 3$.

(3) $f_3 = (x^2 + y, y^2)$, $C(f_3) = \{4xy = 0\}$ is two intersecting lines and $\Delta(f_3) = \{y(y - x^2) = 0\}$ is the sum of a line and a parabola, $\dim O(f_3) = 11$, $\mu(f_3) = 4$.