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Row polynomial matrices of Riordan arrays

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ABSTRACT

Let $R = [r_{n,k}]_{n,k\geq 0}$ be a Riordan array. Define the row polynomials $R_n(q) = \sum_{k=0}^{n} r_{n,k}q^k$ and the row polynomial matrix $R(q) = [r_{n,k}(q)]_{n,k\geq 0}$ by $r_{n,k}(q) = \sum_{j=k}^{n} r_{n,j}q^{j-k}$. Then $R(q)$ is also a Riordan array with the $R_n(q)$ located on the leftmost column of $R(q)$. In this paper we investigate combinatorial properties of the matrix $R(q)$ and the sequence $(R_n(q))_{n\geq 0}$, including their characterizations, the q-total positivity of $R(q)$ and the q-log-convexity of $(R_n(q))_{n\geq 0}$.

1. Introduction

The concept of Riordan arrays was introduced by Shapiro et al. [21]. It turns out to be a very powerful tool in enumerative combinatorics and it is now intensively studied. A *(proper) Riordan array*, denoted by $(d(x), h(x))$, is an infinite lower triangular matrix whose generating function of the kth column is $h^k(x)d(x)$ for $k = 0, 1, 2, \dots$, where

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$d(0) = 1$, $h(0) = 0$ and $h'(0) \neq 0$. It is well known that the set of all Riordan arrays is a group under the matrix multiplication and

$$(d(x), h(x))(g(x), f(x)) = (d(x)g(h(x)), f(h(x))).$$

The identity matrix can be written as $(1, x)$ and the inverse of $(d(x), h(x))$ is given by $(1/d(\bar{h}(x)), \bar{h}(x))$, where $\bar{h}$ is the compositional inverse of h, i.e., $h(\bar{h}(x)) = \bar{h}(h(x)) = x$.

Riordan arrays play an important role in dealing with combinatorial sums (see [13,16,17,23,24] for instance). More precisely, let $R = [r_{n,k}]_{n,k\geq 0}$ be a Riordan array and $(g_k)_{k\geq 0}$ a sequence with the generating function $g(x) = \sum_{k\geq 0} g_k x^k$. Then

$$\sum_{k=0}^{n} r_{n,k} g_k = [x^n]d(x)g(h(x)),$$

which is called the *fundamental theorem of Riordan arrays*. Major interest is usually given to the row sums $\alpha_n = \sum_{k=0}^{n} r_{n,k}$, the alternating row sums $\beta_n = \sum_{k=0}^{n} (-1)^k r_{n,k}$, the weighted row sums $\delta_n = \sum_{k=1}^{n} k r_{n,k}$ and $\gamma_n = \sum_{k=0}^{n} (-1)^k k r_{n,k}$. Define the *row polynomials* of R by

$$R_n(q) = \sum_{j=0}^{n} r_{n,j} q^j, \quad n = 0, 1, 2, \ldots.$$

Then $\alpha_n = R_n(1), \beta_n = R_n(-1), \delta_n = R'_n(1)$ and $\gamma_n = R'_n(-1)$. Let

$$Q = \begin{bmatrix} 1 & & & & \\ q & 1 & & & \\ q^2 & q & 1 & & \\ q^3 & q^2 & q & 1 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Define the *row polynomial matrix* $R(q) = [r_{n,k}(q)]_{n,k\geq 0}$ of R by $R(q) = RQ$. Then

$$r_{n,k}(q) = \sum_{j=k}^{n} r_{n,j} q^{j-k}.$$

Clearly, $R_n(q)$ are located on the leftmost column of $R(q)$, and $R(1)$ is the (partial row) sum matrix of R. Thus the row polynomials and the row polynomial matrix provide a unified approach to deal with certain row sum problems.

In recent years, there have been some papers concerning the row polynomials and the row polynomial matrix of a lower triangular matrix. Wang and Yeh [25] showed that the row polynomials of many well-known triangles have only real zeros. Wang and Yeh [26] gave a sufficient condition that a linear transformation preserves the log-concavity by