Linear Algebra and its Applications 522 (2017) 15–27

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Bounds for the positive and negative inertia index of a graph ☆

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A R T I C L E I N F O

Article history:
Received 2 October 2014
Accepted 3 February 2017
Available online 10 February 2017
Submitted by R. Brualdi

MSC:
05C50

Keywords:
Positive inertia index
Negative inertia index
Matching number
Cyclomatic number

A B S T R A C T

Let G be a graph and let $A(G)$ be adjacency matrix of G. The positive inertia index (respectively, the negative inertia index) of G, denoted by $p(G)$ (respectively, $n(G)$), is defined to be the number of positive eigenvalues (respectively, negative eigenvalues) of $A(G)$. In this paper, we present the bounds for $p(G)$ and $n(G)$ as follows:

$$m(G) - c(G) \le p(G) \le m(G) + c(G),$$

$$m(G) - c(G) \le n(G) \le m(G) + c(G),$$

where $m(G)$ and $c(G)$ are respectively the matching number and the cyclomatic number of G. Furthermore, we characterize the graphs which attain the upper bounds and the lower bounds respectively.

☆ Supported by National Natural Science Foundation of China (11371028), Program for New Century Excellent Talents in University (NCET-10-0001), Scientific Research Fund for Fostering Distinguished Young Scholars of Anhui University (KJJQ1001), Academic Innovation Team of Anhui University Project (KJTD001B).

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http://dx.doi.org/10.1016/j.laa.2017.02.005
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1. Introduction

Let $G = (V(G), E(G)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* $A(G)$ of G is defined to be an $n \times n$ symmetric matrix $[a_{ij}]$ such that $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$ otherwise. The *eigenvalues of* G will be referred to the eigenvalues of $A(G)$. The *positive inertia index* (respectively, the *negative inertia index*) of G, denoted by $p(G)$ (respectively, $n(G)$), is defined to be the number of positive eigenvalues (respectively, negative eigenvalues) of $A(G)$. The *rank* of G, denoted by $r(G)$, is exactly the sum of $p(G)$ and $n(G)$.

According to Hückel theory, the eigenvalues of a chemical graph (i.e. a connected graph with maximum degree at most three) specify the allowed energies of the π molecular orbitals available for occupation by electrons. Such a graph or corresponding molecule is said to be (properly) *closed-shell* if exactly half of its eigenvalues are positive (requiring an even number of vertices), which indicates a stable π-system (see [4]). Chemists are interested in whether the molecular graph of an unsaturated hydrocarbon is (properly) closed-shell, having exactly half of its eigenvalues greater than zero, because this designates a stable electron configuration.

In the mathematics itself, one would like to know or bound $p(G)$ or $n(G)$ for a graph G. The problem is closed related to the *nullity* $\eta(G)$ of G, which is defined to be the number of zero eigenvalues of $A(G)$, since $p(G) + n(G) = |V(G)| - \eta(G)$. Smith [8] proved that a connected graph has exactly one positive eigenvalue if and only if it is complete multipartite. Later Torgašv [9] characterized the graphs with given number of negative eigenvalues. Recently, Yu et al. [11] investigated the minimum positive inertia index among all bicyclic graphs of fixed order with pendant vertices, and characterized the bicyclic graphs with positive index 1 or 2. Ma et al. [6] discussed the positive or the negative inertia index for a graph with at most three cycles, and proved that $|p(G) - n(G)| \le c_1(G)$ for any graph G, where $c_1(G)$ denotes the number of odd cycles contained in G. They conjectured that

$$-c_3(G) \le p(G) - n(G) \le c_5(G), \tag{1.1}$$

where $c_3(G)$ and $c_5(G)$ denote the number of cycles having length 3 modulo 4 and length 1 modulo 4 respectively. In [10] we proved that the conjecture (1.1) holds for line graphs and power trees. In addition, Ma et al. [7] proved that the positive inertia index of the line graph of a tree T lies between the interval $[\frac{\epsilon(T)+1}{2}, \epsilon(T) + 1]$, where $\epsilon(T)$ denotes the number of non-pendant edges of T.

We specify that Daugherty [3] characterized the positive or negative inertia of unicyclic graphs in terms of the matching number; see Theorem 2.6 below. This motivates us to give a characterization for general graphs in terms of the matching number. Denote by $m(G)$ the *matching number* of a graph G, and $c(G)$ the *cyclomatic number* of G defined by $c(G) = |E(G)| - |V(G)| + \theta(G)$, where $\theta(G)$ is the number of connected components