

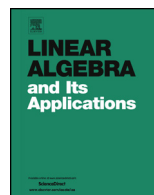


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## On complex matrix scalings of extremal permanent



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## ABSTRACT

A doubly quasi-stochastic (DQS) matrix is said to be maximally (minimally) scaled if it cannot be diagonally scaled to another doubly quasi-stochastic matrix with larger (smaller) permanent. Motivated by a connection to the geometric measure of entanglement of certain symmetric states, we offer a series of results on the structures of the sets of  $n \times n$  maximally scaled ( $MaxSc_n$ ) and minimally scaled ( $MinSc_n$ ) DQS matrices. In particular, we offer a characterization of the set of  $n \times n$  maximally scaled matrices, and use this characterization to show that these matrices form a convex set and that the  $n \times n$  identity matrix is the element of  $MaxSc_n$  with smallest permanent. We then show that real DQS matrices in  $MaxSc_n$  or  $MinSc_n$  must satisfy certain spectral properties, and use these properties to show that all positive definite doubly stochastic matrices are minimally scaled. We finish with a bound on the permanent of any real matrix or Abelian group matrix in  $MinSc_n$ .

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## 1. Introduction and notation

We begin with a few definitions. Recall that a matrix  $A \in \mathbb{C}^{n \times n}$  is said to be positive definite if it is Hermitian with positive eigenvalues. We will make frequent use of the

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decomposition of a positive definite matrix  $M$  into  $M = A + iB$ , where  $A$  and  $B$  are real matrices ( $A$  is symmetric and  $B$  is skew symmetric). In this case, we will denote  $A$  as  $Re(M)$  and  $B$  as  $Im(M)$ . The fundamental type of matrix we will be dealing with are those that are said to be *doubly quasi-stochastic*:

**Definition.** Let  $A$  be a positive definite  $n \times n$  matrix. Then  $A$  is said to be *doubly quasi-stochastic* if the entries in any given row or column sum to 1 (equivalently if  $Ae = e$ , where  $e = (1, 1, \dots, 1)^T$ ). If, in addition,  $A$  has all real, non-negative entries, we say that  $A$  is *doubly stochastic*.

(We should mention at this point that the term “doubly-stochastic” does not usually assume that the matrix in question is positive definite. For the duration of this paper, however, we are only considering positive definite doubly-stochastic matrices, so the above definition will suffice.)

In particular, we will be interested in how we can *scale* a positive definite matrix to a doubly quasi-stochastic matrix:

**Definition.** Let  $A$  be a positive definite  $n \times n$  matrix. We say that an  $n \times n$  diagonal matrix  $D$  *scales*  $A$  if  $B = D^*AD$  is doubly quasi-stochastic. In this case, we say  $B$  is a (*complex*) *scaling* of  $A$ .

We denote the set of all scalings of a particular positive definite matrix  $A$  as  $sc(A)$ . That is,

$$sc(A) = \{B : B \text{ is a doubly quasi-stochastic matrix and } B = D^*AD \\ \text{for some diagonal matrix } D\}.$$

(See [1] for an investigation into the cardinality of  $sc(A)$ .)

Note that if we restrict ourselves to DQS matrices, the set of scaling sets

$$\mathcal{S} = \{sc(A) | A \text{ is a doubly quasi stochastic matrix}\},$$

partitions the DQS matrices into equivalence classes (where the associated equivalence relation is  $A \sim B$  if and only if  $B \in sc(A)$ ).

*Real* matrix scalings have been studied since the early 1960s, when they were first introduced by Sinkhorn in [2], who was interested in their applications to Markov processes. Since then, they have been an area of much study (see [2–4], or [5] for some of the most famous results on real scalings, or [6] for a more recent investigation into scaling symmetric matrices by positive diagonal matrices). This idea was extended further in [7], where the authors discovered that if we allow our matrices to be complex, we obtain an application to the so-called “geometric measure of entanglement” (GME) of certain symmetric states.

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