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On complex matrix scalings of extremal permanent

George Hutchinson

Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario, N1G2W1 Canada

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ABSTRACT

A doubly quasi-stochastic (DQS) matrix is said to be maximally (minimally) scaled if it cannot be diagonally scaled to another doubly quasi-stochastic matrix with larger (smaller) permanent. Motivated by a connection to the geometric measure of entanglement of certain symmetric states, we offer a series of results on the structures of the sets of $n \times n$ maximally scaled ($MaxSc_n$) and minimally scaled ($MinSc_n$) DQS matrices. In particular, we offer a characterization of the set of $n \times n$ maximally scaled matrices, and use this characterization to show that these matrices form a convex set and that the $n \times n$ identity matrix is the element of $MaxSc_n$ with smallest permanent. We then show that real DQS matrices in $MaxSc_n$ or $MinSc_n$ must satisfy certain spectral properties, and use these properties to show that all positive definite doubly stochastic matrices are minimally scaled. We finish with a bound on the permanent of any real matrix or Abelian group matrix in $MinSc_n$.

1. Introduction and notation

We begin with a few definitions. Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is said to be positive definite if it is Hermitian with positive eigenvalues. We will make frequent use of the

E-mail address: ghutch02@uoguelph.ca.

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decomposition of a positive definite matrix M into $M = A + iB$, where A and B are real matrices (A is symmetric and B is skew symmetric). In this case, we will denote A as $Re(M)$ and B as $Im(M)$. The fundamental type of matrix we will be dealing with are those that are said to be *doubly quasi-stochastic*:

Definition. Let A be a positive definite $n \times n$ matrix. Then A is said to be *doubly quasi-stochastic* if the entries in any given row or column sum to 1 (equivalently if $Ae = e$, where $e = (1, 1, ..., 1)^T$). If, in addition, A has all real, non-negative entries, we say that A is *doubly stochastic*.

(We should mention at this point that the term "doubly-stochastic" does not usually assume that the matrix in question is positive definite. For the duration of this paper, however, we are only considering positive definite doubly-stochastic matrices, so the above definition will suffice.)

In particular, we will be interested in how we can *scale* a positive definite matrix to a doubly quasi-stochastic matrix:

Definition. Let A be a positive definite $n \times n$ matrix. We say that an $n \times n$ diagonal matrix D *scales* A if $B = D^*AD$ is doubly quasi-stochastic. In this case, we say B is a *(complex) scaling* of A.

We denote the set of all scalings of a particular positive definite matrix A as $sc(A)$. That is,

$$sc(A) = \{B : B \text{ is a doubly quasi-stochastic matrix and } B = D^*AD \\ \text{for some diagonal matrix } D\}.$$

(See [1] for an investigation into the cardinality of $sc(A)$.)

Note that if we restrict ourselves to DQS matrices, the set of scaling sets

$$\mathcal{S} = \{sc(A) | A \text{ is a doubly quasi stochastic matrix}\},$$

partitions the DQS matrices into equivalence classes (where the associated equivalence relation is $A \sim B$ if and only if $B \in sc(A)$).

Real matrix scalings have been studied since the early 1960s, when they were first introduced by Sinkhorn in [2], who was interested in their applications to Markov processes. Since then, they have been an area of much study (see [2–4], or [5] for some of the most famous results on real scalings, or [6] for a more recent investigation into scaling symmetric matrices by positive diagonal matrices). This idea was extended further in [7], where the authors discovered that if we allow our matrices to be complex, we obtain an application to the so-called "geometric measure of entanglement" (GME) of certain symmetric states.