Linear Algebra and its Applications 522 (2017) 153–160

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The Strong Arnold Property for 4-connected flat graphs

Alexander Schrijver *, Bart Sevenster

Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Netherlands

ARTICLE INFO

Article history:
Received 2 December 2015
Accepted 2 February 2017
Available online 14 February 2017
Submitted by R. Brualdi

MSC:
05C50
15A18
05C10

Keywords:
Flat graph
Colin de Verdière parameter
Strong Arnold Property

ABSTRACT

We show that if $G = (V, E)$ is a 4-connected flat graph, then any real symmetric $V \times V$ matrix M with exactly one negative eigenvalue and satisfying, for any two distinct vertices i and j, $M_{ij} < 0$ if i and j are adjacent, and $M_{ij} = 0$ if i and j are nonadjacent, has the Strong Arnold Property: there is no nonzero real symmetric $V \times V$ matrix X with $MX = 0$ and $X_{ij} = 0$ whenever i and j are equal or adjacent. (A graph G is *flat* if it can be embedded injectively in 3-dimensional Euclidean space such that the image of any circuit is the boundary of some disk disjoint from the image of the remainder of the graph.)
This applies to the Colin de Verdière graph parameter, and extends similar results for 2-connected outerplanar graphs and 3-connected planar graphs.

1. Introduction

Let $G = (V, E)$ be an undirected graph. Call a real symmetric $V \times V$ matrix a *well-signed G-matrix* if for all distinct $i, j \in V$: $M_{ij} < 0$ if $ij \in E$ and $M_{ij} = 0$ if $ij \notin E$.

* Corresponding author.
E-mail address: lex@cwi.nl (A. Schrijver).

http://dx.doi.org/10.1016/j.laa.2017.02.002
0024-3795/© 2017 Elsevier Inc. All rights reserved.

(No condition on the diagonal elements.) For any real symmetric matrix M, let $\lambda^-(M)$ be the number of negative eigenvalues, taking multiplicities into account. The *corank* of M is the dimension of its nullspace $\ker(M)$.

The famous *Colin de Verdière parameter* $\mu(G)$ [1] is defined to be the maximal corank of any well-signed G-matrix M with $\lambda^-(M) = 1$ and having the *Strong Arnold Property*:

(1) there is no nonzero real symmetric $V \times V$ matrix X with $MX = 0$ and $X_{ij} = 0$ whenever i and j are equal or adjacent.

The interest of the parameter $\mu(G)$ was exhibited by Colin de Verdière [1], who showed that $\mu(G)$ is *minor-monotone*, that is, $\mu(H) \leq \mu(G)$ if H is a minor of G, — in other words, for each k, the collection of graphs G with $\mu(G) \leq k$ is closed under taking minors; hence there are finitely many forbidden minors, by Robertson and Seymour [9]. The Strong Arnold Property is crucial for the minor-monotonicity.

Moreover, Colin de Verdière [1] showed (i), (ii), and (iii) in:

(2) (i) $\mu(G) \leq 1$ if and only if G is a disjoint union of paths,
(ii) $\mu(G) \leq 2$ if and only if G is outerplanar,
(iii) $\mu(G) \leq 3$ if and only if G is planar,
(iv) $\mu(G) \leq 4$ if and only if G is flat.

Statement (iv) was proved by Robertson, Seymour, and Thomas [10] (only if) and Lovász and Schrijver [6] (if). Recall that a graph G is *flat* if it can be embedded injectively in $\mathbb{R}^3$ such that the image of any circuit is the boundary of some disk disjoint from the image of the remainder of the graph. As was shown in [10], a graph is flat if and only if it is linklessly embeddable, that is, can be embedded injectively in $\mathbb{R}^3$ such that the images of any two disjoint circuits are unlinked. We refer to [4] for a survey of the Colin de Verdière parameter.

A short proof of (iii) was given by van der Holst [2], which proof also implies that if G is 3-connected and planar, then any well-signed G-matrix M with $\lambda^-(M) = 1$, has corank at most 3. So the Strong Arnold Property is not needed to define $\mu(G)$ for such graphs G. That is, if we define $\kappa(G)$ to be the maximum corank of any well-signed G-matrix M with $\lambda^-(M) = 1$, then $\kappa(G) = \mu(G)$ for 3-connected planar graphs G. Here 3-connectivity cannot be relaxed to 2-connectivity, since $\kappa(K_{2,t}) = t$ for all t, while $\mu(K_{2,t}) = 3$ for all $t \geq 3$. In [6], it was shown that $\kappa(G) = \mu(G)$ also for 4-connected flat graphs.

The latter means that for any 4-connected flat graph G, among the well-signed G-matrices M with $\lambda^-(M) = 1$ that maximize $\text{corank}(M)$, there is one having the Strong Arnold Property. In this paper, we prove that for any 4-connected flat graph G, *each* well-signed G-matrix M with $\lambda^-(M) = 1$ has the Strong Arnold Property. This extends results of van der Holst [3] who proved this for 2-connected outerplanar graphs and for 3-connected planar graphs. In fact, one may show that if this holds for all