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Nonoscillation of second-order linear difference systems with varying coefficients

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ABSTRACT

This paper deals with nonoscillation problem about the non-autonomous linear difference system

$$\mathbf{x}_n = A_n \mathbf{x}_{n-1}, \qquad n = 1, 2, \ldots,$$

where A_n is a 2×2 variable matrix that is nonsingular for $n \in \mathbb{N}$. In the special case that A is a constant matrix, it is well-known that all non-trivial solutions are nonoscillatory if and only if all eigenvalues of A are positive real numbers; namely, $\det A > 0$, $\mathrm{tr} A > 0$ and $\det A/(\mathrm{tr} A)^2 \leq 1/4$. The well-known result can be said to be an analogy of ordinary differential equations. The results obtained in this paper extend this analogy result. In other words, this paper clarifies the distinction between difference equations and ordinary differential equations. Our results are explained with some specific examples. In addition, figures are attached to facilitate understanding of those examples.

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1. Introduction

We consider the second-order linear time-variant system

$$\mathbf{x}_n = A_n \mathbf{x}_{n-1}, \qquad n = 1, 2, \dots, \tag{1.1}$$

where

$$\mathbf{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{and} \quad A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

in which the components x_n and y_n and the coefficients a_n, b_n, c_n and d_n are real numbers. It is always assumed that the matrix A_n is nonsingular for $n \in \mathbb{N}$. Needless to say, equation (1.1) has the trivial solution $\{\mathbf{x}_n\}$; that is, $(x_n, y_n) = (0, 0)$ for $n \in \mathbb{N}$. A non-trivial solution $\{\mathbf{x}_n\}$ of (1.1) is said to be *oscillatory with respect to the first* (resp., *second*) *component* if, for every $n \in \mathbb{N}$ there exists an $m \geq n$ such that $x_m x_{m+1} \leq 0$ (resp., $y_m y_{m+1} \leq 0$). Otherwise, it is said to be *nonoscillatory with respect to the first* (or *second*) *component*. Hence, if a non-trivial solution $\{\mathbf{x}_n\}$ of (1.1) is nonoscillatory with respect to the first (resp., second) component, then there exists an $m \in \mathbb{N}$ such that $x_n > 0$ for $n \geq m$ or $x_n < 0$ for $n \geq m$ (resp., $y_n > 0$ for $n \geq m$ or $y_n < 0$ for $n \geq m$). It is clear that if $\{\mathbf{x}_n\}$ is a solution of (1.1), then $\{-\mathbf{x}_n\}$ is also a solution of (1.1). Hence, we can assume without loss of generality that a non-trivial solution $\{\mathbf{x}_n\}$ of (1.1) which is nonoscillatory with respect to the first (resp., second) component satisfy that x_n (resp., y_n) is positive for all large n. A non-trivial solution $\{\mathbf{x}_n\}$ of (1.1) is said to be *nonoscillatory* if it is nonoscillatory with respect to the first and second components.

The purpose of this paper is to give sufficient conditions for all non-trivial solutions of (1.1) to be nonoscillatory with respect to the first (or second) component. Of course, the coefficients of the matrix A_n determine whether or not all non-trivial solutions of (1.1) are nonoscillatory with respect to the first component.

In the special case that

$$A_n \equiv A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c and d are real constants, system (1.1) is equivalent to the second-order autonomous linear equations

$$x_{n+1} + (\det A) x_{n-1} = (\operatorname{tr} A) x_n \tag{1.2}$$

and

$$y_{n+1} + (\det A) y_{n-1} = (\operatorname{tr} A) y_n \tag{1.3}$$

for $n \in \mathbb{N}$. It is clear that if $\det A < 0$, then the characteristic equation