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Projective cyclic groups in higher dimensions ☆

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In this article we provide a classification of the projective transformations in $PSL(n+1,\mathbb{C})$ considered as automorphisms of the complex projective space $\mathbb{P}^n_{\mathbb{C}}$. Our classification is an interplay between algebra and dynamics. Just as in the case of isometries of $CAT(0)$-spaces, this is given by means of three types of transformations, namely: elliptic, parabolic and loxodromic. We describe the dynamics in each case, more precisely we determine the corresponding Kulkarni's limit set, the equicontinuity region, minimal sets, the discontinuity region and maximal regions where the corresponding cyclic group acts properly discontinuously. We also provide, in each case, some equivalent ways to classify the projective transformations.

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0. Introduction

Discrete groups of projective transformations arise as monodromy groups of ordinary differential equations (see [14]), associated to Ricatti's foliations (see [18]) or as the monodromy groups of the so called orbifold uniformizing differential equations (see [20]). However outside the groups coming from complex hyperbolic geometry, little is known about their dynamics, see Chapters 9 and 10 in [4]. Yet, as in the one dimensional case, one might expect interesting results. In this paper we deal with the basic problem of classifying the projective transformations and understand their dynamics.

When we look at elements in $\mathrm{PU}(1,n)$, one has that they preserve a ball, then, as in the one dimensional case, this fact enables us to classify the transformations in $\mathrm{PU}(1,n)$ by means of their fixed points and their position in the closed complex ball. More precisely, an element is said to be: elliptic if it has a fixed point in the complex ball, parabolic if it has a unique fixed point in the boundary of the complex ball and finally the element is said to be loxodromic if it has exactly two fixed points in the boundary of the complex ball. However, when we deal with automorphisms of $\mathbb{P}^n_{\mathbb{C}}$, this type of classification makes no sense, since in general there is not an invariant ball. So, to extend the previous classification to $PSL(n+1,\mathbb{C})$, we must think dynamically, more precisely we must look into the local behavior around the fixed points. The following definition captures this point of view.

Definition 0.1. Let $\gamma \in PSL(n+1,\mathbb{C})$ be a projective transformation, then

1. The element γ is called elliptic if there is a lift $\widetilde{\gamma} \in SL(n+1,\mathbb{C})$ of γ, which is diagonalizable and each of its eigenvalues is unitary.
2. The element γ is called loxodromic is there is a lift $\widetilde{\gamma} \in SL(n+1,\mathbb{C})$ of γ having non-unitary eigenvalue.
3. The element γ is parabolic, if there is a lift $\widetilde{\gamma} \in SL(n+1,\mathbb{C})$ of γ which is non-diagonalizable and has only unitary eigenvalues.

Clearly this definition exhausts all the possibilities and agrees with the standard classification in the one and two dimensional setting, as well as, in the case of transformations in $PU(1,n)$, $n \geq 1$, see Chapter 7 in [4] and Chapter 6 in [10]. There is also another classification of the projective transformations of $PSL(3,\mathbb{C})$ in terms of the fixed set given in [18], which is closely related to our classification but properly speaking does not agree with the one exposed here.

As in the two dimensional case, see Chapter 6 in [4], we want to know the relations between different notions of limit set (Kulkarni's limit set, complement of the discontinuity set, complement of the equicontinuity set, minimal sets, complements of maximal sets where the action is properly discontinuously) for complex Kleinain groups in all dimensions, and a nice starting point towards the solution of this problem is providing a description of the following sets for cyclic groups. In this article we show: