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Congruences for permanents and determinants of circulants

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ABSTRACT

Starting from particular congruences concerning permanents of some (0, 1) circulant matrices, we derive more general congruences for permanents and determinants. We also analyze the relation of such results with some congruences satisfied by the norms of algebraic integers in cyclotomic fields.

1. Introduction

Permanents and determinants of (0, 1) circulant matrices have been recently studied in several works. Determinants of such matrices, when the number of 1's per row is limited, are known (in view of results in [2]) to be computable in any case in $O(n)$ time (n denoting the matrix size), while presently the calculation of permanents requires $O(n2^n)$ time (via Ryser's algorithm [4]) for generic (0, 1) circulants having at least four ones per row; only in the case of (0, 1) circulants with three ones per row the permanent

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can be calculated in $O(n)$ time, in view of appropriate formulas (shown in [1]) giving the permanent as a linear combination of four determinants of suitable related matrices.

In [5] the permanents of $n \times n$ (0, 1)-circulant matrices were investigated from an algebraic viewpoint, obtaining constraining congruences for their values for particular classes of n's. In this paper we first show that similar congruences hold for both permanents and determinants and for a more general class of n's. After deriving such extended relations, we combine some of them with known formulas for determinants of circulants; such formulas involve appropriate complex unity roots. In this way we obtain interesting connections between determinants of (0, 1) circulants and norms of some algebraic integers in cyclotomic fields.

After this, extending the discussion to generic (not necessarily (0, 1)) $n \times n$ circulant matrices with entries in $\mathbf{Z}$, we derive more general relations for permanents and determinants. The connection, existing between the extended results on determinants and the norms of algebraic integers in cyclotomic fields, is also shown.

2. Permanents and determinants of (0, 1) circulants

Let Σ_n denote the set of all permutations of $\mathbf{Z}_n$ (or, equivalently, of permutations of the first n positive integers). Given an $n \times n$ square matrix A, the permanent of A is the number

$$\mathrm{Per}(A) = \sum_{\sigma \in \Sigma_n} \prod_{i=1}^{n} a_{i,\,\sigma(i)}\,,$$

while the determinant of A is

$$\mathrm{Det}(A) = \sum_{\sigma \in \Sigma_n} (-1)^{|\sigma|} \prod_{i=1}^{n} a_{i,\,\sigma(i)}\,.$$

In the rest of the paper we denote, respectively, by I_n the $n \times n$ identity matrix and by P_n the $n \times n$ circulant (0, 1) matrix with the only 1's in positions $(n,\,1)$ and $(i,\,i+1)$, $i = 1,\,2,\,...,\,n-1$.

For generic number field K, the notation $N_{\mathbf{Q}}^{K}$ indicates the classical norm function defined over K.

Now we fix a generic $n \times n$ (0, 1) circulant matrix A with k non-zero entries per row; A can be expressed as $P_n^{i_1} + P_n^{i_2} + \ldots + P_n^{i_k}$, where $0 \le i_1 < i_2 < \cdots < i_k \le n-1$.

We observe that the permanent of A (or, equivalently, the number of nonzero addenda in the expression of the determinant of A) is equal to the cardinality of the set $E = E_A = E_{i_1,\,i_2,\,...,\,i_k}$ of permutations $\pi \in \Sigma_n$ such that, for each $x \in \mathbf{Z}_n$, $\pi(x) \in \{x + i_1,\, x+i_2,\, ...,\, x+i_k\}$ (mod n). For any permutation $\pi \in \Sigma_n$, let us define the increment function $T_\pi : \mathbf{Z}_n \to \mathbf{Z}_n$ by setting