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Optimal exponents for Hardy–Littlewood inequalities for m-linear operators ☆

R.M. Aron [a], D. Núñez-Alarcón [a,b], D.M. Pellegrino [c], D.M. Serrano-Rodríguez [a,b,*]

[a] Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA
[b] Departamento de Matemática, Universidade Federal de Pernambuco, 50.740-560, Recife, Brazil
[c] Departamento de Matemática, Universidade Federal da Paraíba, 58.051-900, João Pessoa, Brazil

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The Hardy–Littlewood inequalities on ℓ_p spaces provide optimal exponents for some classes of inequalities for bilinear forms on ℓ_p spaces. In this paper we investigate in detail the exponents involved in Hardy–Littlewood type inequalities and provide several optimal results that were not achieved by the previous approaches. Our first main result asserts that for $q_1, ..., q_m > 0$ and an infinite-dimensional Banach space Y attaining its cotype $\cot Y$, if

$$\frac{1}{p_1} + ... + \frac{1}{p_m} < \frac{1}{\cot Y},$$

then the following assertions are equivalent:
(a) There is a constant $C^Y_{p_1,...,p_m} \geq 1$ such that

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* Corresponding author.
E-mail addresses: aron@math.kent.edu (R.M. Aron), danielnunezal@gmail.com (D. Núñez-Alarcón), dmpellegrino@gmail.com, pellegrino@pq.cnpq.br (D.M. Pellegrino), dmserrano0@gmail.com (D.M. Serrano-Rodríguez).

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$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_m=1}^{\infty}\|A(e_{j_1},\ldots,e_{j_m})\|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$
$$\leq C^Y_{p_1,\ldots,p_m}\|A\|$$

for all continuous m-linear operators $A:\ell_{p_1}\times\cdots\times\ell_{p_m}\to Y$.
(b) The exponents $q_1,\ldots,q_m$ satisfy

$$q_1\geq\lambda^{p_1,\ldots,p_m}_{m,\cot Y},\, q_2\geq\lambda^{p_2,\ldots,p_m}_{m-1,\cot Y},\ldots,q_m\geq\lambda^{p_m}_{1,\cot Y},$$

where, for $k=1,\ldots,m$,

$$\lambda^{p_k,\ldots,p_m}_{m-k+1,\cot Y}:=\frac{\cot Y}{1-\left(\frac{1}{p_k}+\ldots+\frac{1}{p_m}\right)\cot Y}.$$

As an application of the above result we generalize one of the classical Hardy–Littlewood inequalities for bilinear forms to the m-linear setting. Our result is sharp in a very strong sense: the constants and exponents are optimal, even if we consider mixed sums.

1. Introduction

Let $\mathbb{K}$ be the real or complex scalar field. In 1934 Hardy and Littlewood proved three theorems (Theorems 1.1, 1.2, 1.3, below) on the summability of bilinear forms on $\ell_p\times\ell_q$ (here, and henceforth, when $p=\infty$ we consider c_0 instead of ℓ_∞). For any function f we shall consider $f(\infty):=\lim_{s\to\infty}f(s)$ and for any $s\geq1$ we denote the conjugate index of s by s^*, i.e., $\frac{1}{s}+\frac{1}{s^*}=1$.

For all $p,q\in(1,\infty]$, such that $\frac{1}{p}+\frac{1}{q}<1$, let us define

$$\lambda:=\frac{pq}{pq-p-q},$$

and

$$\mu=\frac{4pq}{3pq-2p-2q}.$$

If p and q are simultaneously ∞, then λ and μ are 1 and 4/3 respectively.

From now on, $(e_k)_{k=1}^{\infty}$ denotes the sequence of canonical vectors in ℓ_p.

Theorem 1.1. *(See Hardy and Littlewood [12, Theorem 1].) Let $p,q\in[2,\infty]$, with $\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}$. There is a constant $C_{p,q}\geq1$ such that*