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Indecomposable modules of a family of solvable Lie algebras

Paolo Casati [a], Andrea Previtali [a], Fernando Szechtman [b,*]

[a] *Dipartimento di Matematica e Applicazioni, University of Milano-Bicocca, Milano, Italy*
[b] *Department of Mathematics and Statistics, University of Regina, Canada*

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We classify all uniserial modules of the solvable Lie algebra $\mathfrak{g} = \langle x \rangle \ltimes V$, where V is an abelian Lie algebra over an algebraically closed field of characteristic 0 and x is an arbitrary automorphism of V.

1. Introduction

Let F be an algebraically closed field of characteristic 0. All vector spaces, including all Lie algebras and their modules, are assumed to be finite dimensional over F.

Recall that a module is said to be indecomposable if it cannot be decomposed as the direct sum of two non-trivial submodules. Naturally, knowing all indecomposable

* Corresponding author.
E-mail addresses: paolo.casati@unimib.it (P. Casati), andrea.previtali@unimib.it (A. Previtali), fernando.szechtman@gmail.com (F. Szechtman).

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modules of a given Lie algebra would provide a complete description of all its modules. Unfortunately, the problem of classifying all indecomposable modules of a given Lie algebra – that is not semisimple or one-dimensional – is virtually unsolvable, even in the case of the two-dimensional abelian Lie algebra, as observed in a celebrated paper by Gelfand and Ponomarev [10].

In spite of this fact, many types of indecomposable modules of non-semisimple Lie algebras have been recently classified, see for example [2–9,11–13].

In all these papers the central idea is to consider particular classes of indecomposable modules for which a complete classification can be achieved. Besides the irreducible modules, the simplest type of indecomposable module is, in a certain sense, the uniserial one. This is a module having a unique composition series, i.e. a non-zero module whose submodules form a chain. Alternatively, such modules can be defined as follows.

Let $\mathfrak{g}$ be a given Lie algebra and let U be a non-zero $\mathfrak{g}$-module. The socle series

$$0 = \mathrm{soc}_0(U) \subset \mathrm{soc}_1(U) \subset \cdots \subset \mathrm{soc}_k(U) = U$$

of U is inductively defined by declaring $\mathrm{soc}_i(U)/\mathrm{soc}_{i-1}(U)$ to be the socle of $U/\mathrm{soc}_{i-1}(U)$, that is, the sum of all irreducible submodules of $U/\mathrm{soc}_{i-1}(U)$, for $1 \leq i \leq k$. Then U is uniserial if and only if the socle series of U has irreducible factors.

In the last years, the classification of the uniserial modules of important classes of solvable and perfect Lie algebras has been achieved in various research papers [1,2,4,5, 12]. In particular, [12] and [1] classify a wider class of modules, called cyclic in [12] and perfect cyclic in [1], over the perfect Lie algebras $\mathfrak{sl}(2) \ltimes F^2$ and $\mathfrak{sl}(n+1) \ltimes F^{n+1}$, for $F = \mathbb{C}$, respectively.

The aim of this paper is to proceed further in the study of uniserial modules. We shall, indeed, classify the uniserial modules of a distinguished class of solvable Lie algebras, namely those of the form $\mathfrak{g} = \langle x \rangle \ltimes V$, where V is an abelian Lie algebra and x is an arbitrary automorphism of V.

A proper ideal $\mathfrak{a}$ of $\mathfrak{g}$ is of the form $\mathfrak{a} = W$, where W is an x-invariant subspace of V. Thus, either $W = V$ and $\mathfrak{g}/\mathfrak{a} \cong \langle x \rangle$ is one-dimensional, or else $\mathfrak{g}/\mathfrak{a} \cong \langle \overline{x} \rangle \ltimes \overline{V}$, where $\overline{V} = V/W \neq (0)$ and $\overline{x}$ is the automorphism that x induces on $\overline{V}$.

We know from [4] all uniserial modules over an abelian Lie algebra as well as all uniserial $\mathfrak{g}$-modules when x is diagonalizable. Thus, it suffices to classify all *faithful* uniserial $\mathfrak{g}$-modules when x is not diagonalizable. In this regard, our main results are as follows.

In §2 we construct a family of non-isomorphic faithful uniserial representations of $\mathfrak{g}$ when x acts on V via a single Jordan block of size $n > 1$. This family consists of all matrix representations

$$R_{\alpha,k,X} \to \mathfrak{gl}(n+1),\ R_{\alpha,n} \to \mathfrak{gl}(n+1),\ R_{\alpha,1} \to \mathfrak{gl}(n+1), \tag{1.1}$$

where