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General explicit descriptions for intertwining operators and direct rotations of two orthogonal projections ☆

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ABSTRACT

In this paper, based on the block operator technique and operator spectral theory, general explicit descriptions for intertwining operators and direct rotations of two orthogonal projections are established. As a consequence, Kato's result (Kato, 1996 [14]) is improved, so are J. Avron, R. Seiler and B. Simon's Theorem 2.3 (Avron et al., 1994 [6]) and C. Davis, W.M. Kahan's result (Davis and Kahan, 1970 [11]).

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$. An operator P is called an orthogonal projection if $P = P^* = P^2$. Let $\mathcal{P}$ be the set of

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all orthogonal projections in $\mathcal{B}(\mathcal{H})$. As well-known, orthogonal projections on a Hilbert space are basic objects of study in operator theory (see [1–20] and therein references). In this paper, we will pay attention to the characterization of intertwining operators and direct rotations of two orthogonal projections. Let the set of all unitaries in $\mathcal{B}(\mathcal{H})$ be denoted by $\mathcal{U}(\mathcal{H})$. If P and Q are orthogonal projections and there exists a unitary $U \in \mathcal{U}(\mathcal{H})$ such that

$$UP = QU, \tag{1}$$

then U is called an outer intertwining operator of P and Q. The set of all outer intertwining operators of a pair (P, Q) of orthogonal projections is denoted by

$$\text{out}\, \mathcal{U}_Q(P).$$

Similarly, if

$$PU = UQ, \tag{2}$$

then U is called an inner intertwining operator of P and Q. The set of all inner intertwining operators of a pair (P, Q) of orthogonal projections is denoted by

$$\text{inn}\, \mathcal{U}_Q(P).$$

Moreover, if both of

$$PU = UQ \text{ and } UP = QU \tag{3}$$

hold, then U is called an intertwining operator of P and Q. The set of all intertwining operators of a pair (P, Q) of orthogonal projections is denoted by

$$\text{int}\, \mathcal{U}_Q(P).$$

Fix a pair (P, Q) of orthogonal projections. A unitary $U \in \mathcal{U}(\mathcal{H})$ is called a direct rotation from P to Q (see [1] and [11]) if

$$UP = QU, \quad U^2 = (Q^\perp - Q)(P^\perp - P), \quad \text{Re}U \geq 0, \tag{4}$$

where $K^\perp = I - K$ if K is an orthogonal projection.

If P and Q are orthogonal projections with $\| P - Q \| < 1$, Kato in [14] verified that there exists $U \in \mathcal{U}(\mathcal{H})$ such that $PU = UQ$. Moreover, Avron, Seiler and Simon ([6]) proved that if P and Q are orthogonal projections on $\mathcal{H}$ with $\| P - Q \| < 1$, then there exists a unitary $U \in \mathcal{U}(\mathcal{H})$ with $UPU^* = Q, UQU^* = P$. If P and Q are orthogonal projections having no common eigenvectors, the main result shown by Amrein, Sinha ([2])