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Addendum

Addendum to "Lattices from equiangular tight frames" [Linear Algebra Appl. 510 (2016) 395–420]

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ABSTRACT

This note supplies additions to our paper quoted in the title on integral spans of tight frames in Euclidean spaces. In that previous paper, we considered the case of an equiangular tight frame (ETF), proving that if its integral span is a lattice then the frame must be rational, but overlooking a simple argument in the reverse direction. Thus our first result here is that the integral span of an ETF is a lattice if and only if the frame is rational. Further, we discuss conditions under which such lattices are eutactic and perfect and, consequently, are local maxima of the packing density function in the dimension of their span. In particular, the unit $(276, 23)$ equiangular tight frame is shown to be eutactic and perfect. More general tight frames and their norm-forms are considered as well, and definitive results are obtained in dimensions two and three.

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1. Introduction and main results

We denote by $\langle\ ,\ \rangle$ the usual inner product on $\mathbb{R}^k$ and by $\|\boldsymbol{x}\| := \langle \boldsymbol{x}, \boldsymbol{x}\rangle^{1/2}$ the Euclidean norm. Let $n \geq k$ and let $\mathcal{F} := \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\} \subset \mathbb{R}^k$ be a set of vectors such that $\operatorname{span}_{\mathbb{R}} \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\} = \mathbb{R}^k$. Put

$$\Lambda(\mathcal{F}) = \operatorname{span}_{\mathbb{Z}} \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\}.$$

We consider $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$ as column vectors and denote by G the $k \times n$ matrix with these vectors as columns. Clearly, we may think of $\Lambda(\mathcal{F})$ as the set $\{G\boldsymbol{a} : \boldsymbol{a} \in \mathbb{Z}^n\}$. The *norm-form* associated with $\mathcal{F}$ is the quadratic form

$$Q_{\mathcal{F}}(\boldsymbol{a}) = \|G\boldsymbol{a}\|^2 = \langle G^\top G\boldsymbol{a}, \boldsymbol{a}\rangle.$$

A quadratic form $Q(\boldsymbol{a}) = \langle H\boldsymbol{a}, \boldsymbol{a}\rangle$ with a symmetric matrix H is said to *have separated values* if the values of $Q(\boldsymbol{a})$ for $\boldsymbol{a}$ in $\mathbb{Z}^n$ are separated by a positive number, that is, if

$$\inf\{|Q(\boldsymbol{a}) - Q(\boldsymbol{b})| : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n, Q(\boldsymbol{a}) \neq Q(\boldsymbol{b})\} > 0.$$

We call the set $\mathcal{F}$ *rational* if the inner products $\langle \boldsymbol{f}_i, \boldsymbol{f}_j\rangle$ are rational numbers for all $1 \leq i, j \leq n$. This is equivalent to saying that the entries of the $n \times n$ Gram matrix $G^\top G$ are all rational.

We here study two questions. First, we are interested in conditions ensuring that $\Lambda(\mathcal{F})$ is a lattice, and if it is, in properties of this lattice. Secondly, we look for conditions ensuring that $Q_{\mathcal{F}}$ has separated values. Recall that a *lattice* is a discrete additive subgroup of $\mathbb{R}^k$. The set $\Lambda(\mathcal{F})$ of the integer linear combinations of the vectors $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$ is clearly an additive subgroup of $\mathbb{R}^k$, but it may contain accumulation points and hence not be discrete, which would prevent it from being a lattice. Obviously, $\Lambda(\mathcal{F})$ is a lattice if and only if 0 is not an accumulation point of the values of $Q_{\mathcal{F}}$, that is, if and only if there exists an $\varepsilon > 0$ such that $Q_{\mathcal{F}}(\boldsymbol{a}) \notin (0, \varepsilon)$ for $\boldsymbol{a} \in \mathbb{Z}^n$. A lattice $\Lambda \subset \mathbb{R}^k$ is said to be of full rank if $\operatorname{span}_{\mathbb{R}} \Lambda = \mathbb{R}^k$. As we always suppose that $\operatorname{span}_{\mathbb{R}}\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\} = \mathbb{R}^k$, the set $\Lambda(\mathcal{F})$ is a lattice if and only if it is a full-rank lattice.

For further reference, we state the following simple observation, which, unfortunately, was overlooked in [2].

Proposition 1. *If $\mathcal{F}$ is rational, then $\Lambda(\mathcal{F})$ is a lattice and the values of $Q_{\mathcal{F}}$ are separated.*

Proof. There is an integer $d > 0$ such that all entries of $dG^\top G$ are integers, and hence $dQ_{\mathcal{F}}(\boldsymbol{a}) = d\langle G^\top G\boldsymbol{a}, \boldsymbol{a}\rangle$ assumes values in $\{0, 1, 2, \ldots\}$ for $\boldsymbol{a} \in \mathbb{Z}^n$, which shows that $Q_{\mathcal{F}}$ has separated values and does not take values in $(0, 1/d)$. $\square$

Clearly, if $\mu \neq 0$ is any real number, then $\Lambda(\mu\mathcal{F})$ is also a lattice and the values of $Q_{\mu\mathcal{F}}$ are separated.

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