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An arithmetic–geometric mean inequality for singular values and its applications

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ABSTRACT

In this short note, we give a new equivalent form of the arithmetic–geometric mean inequality for singular values. As applications of our result, we give a new proof of an inequality due to Bhatia and Davis (1993) [4] and we obtain a singular value inequality for matrix means, which is similar to one proved by Drury (2012) [9]. Finally, we present a log-majorization inequality for singular values.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. We shall always denote the singular values of A by $s_1(A) \geq \cdots \geq s_n(A) \geq 0$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. If $A \in M_n$ has real eigenvalues, we label them as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$.

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Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. Let $A,\ B \in M_n$ be positive definite, the geometric mean of A and B, denoted by $A\#B$, is defined as

$$A\#B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}.$$

As pointed out in [3, p. 107] this definition could be extended to positive semidefinite matrices $A,\ B$ by a limit from above:

$$A\#B = \lim_{\varepsilon\to 0}\left[(A+\varepsilon I)^{1/2}\left((A+\varepsilon I)^{-1/2}(B+\varepsilon I)(A+\varepsilon I)^{-1/2}\right)^{1/2}(A+\varepsilon I)^{1/2}\right].$$

Let A and B be positive semidefinite. The well-known arithmetic–geometric mean inequality for singular values due to Bhatia and Kittaneh [6, Theorem 1] says that

$$s_j(AB) \le \frac{1}{2}s_j\left(A^2+B^2\right),\ \ j=1,\cdots,n, \tag{1.1}$$

which can be stated in another form: If $A, B \in M_n$, then

$$s_j(A^*B) \le \frac{1}{2}s_j(AA^*+BB^*),\ \ j=1,\cdots,n. \tag{1.2}$$

For more information on this inequality and its equivalent forms the reader is referred to [1] and the references therein.

Let $A, X, B \in M_n$. Bhatia and Davis [4, Theorem 1] proved that

$$\|A^*XB\| \le \frac{1}{2}\|AA^*X+XBB^*\|. \tag{1.3}$$

This is the arithmetic–geometric mean inequality for unitarily invariant norms.

Let A and B be positive semidefinite. Recently, Drury [9, Theorem 1] proved that

$$s_j(AB) \le \frac{1}{4}s_j(A+B)^2,\ \ j=1,\cdots,n, \tag{1.4}$$

which is a question posed by Bhatia and Kittaneh [7, Inequality (1.7)] and it is a strengthening of inequality (1.1). For more information on singular value and norm inequalities related to matrix means the reader is referred to [2,3].

In this short note, we first present a new equivalent form of (1.2). As applications of our result, we give a new proof of inequality (1.3) and we obtain a singular value inequality for matrix means, which is similar to (1.4). Finally, we present a log-majorization inequality for singular values.

2. Main results

Now, we show a new equivalent form of the arithmetic–geometric mean inequality for singular values.