ARTICLE IN PRESS LAA:13592

Linear Algebra and its Applications ••• (••••) •••–•••

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On Drury's solution of Bhatia & Kittaneh's question

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ARTICLE INFO

Article history:
Received 4 September 2015
Accepted 18 February 2016
Available online xxxx
Submitted by P. Semrl

Dedicated to Rajendra Bhatia on the occasion of his 65th birthday

MSC:
15A45
15A60

Keywords:
AM–GM inequality
Singular value
Eigenvalue

ABSTRACT

Let A, B be $n \times n$ positive semidefinite matrices. Bhatia and Kittaneh asked whether it is true

$$\sqrt{\sigma_j(AB)} \le \frac{1}{2}\lambda_j(A+B), \qquad j = 1, \dots, n$$

where $\sigma_j(\cdot)$, $\lambda_j(\cdot)$, is the j-th largest singular value, eigenvalue, respectively. The question was recently solved by Drury in the affirmative. This article revisits Drury's solution. In particular, we simplify the proof for a key auxiliary result in his solution.

1. Introduction

Bhatia has made many fundamental contributions to Matrix Analysis [2]. One of his favorite topics is matrix inequalities. Roughly speaking, matrix inequalities are non-commutative versions of the corresponding scalar inequalities. To get a glimpse of

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http://dx.doi.org/10.1016/j.laa.2016.02.025

this topic, let us start with a simple example. The simplest AM–GM inequality says that

$$a, b > 0 \implies \frac{a+b}{2} \geq \sqrt{ab}.$$

Now it is known that [3, p. 107] its most "direct" noncommutative version is

$$A, B \text{ are } n \times n \text{ positive definite matrices} \implies \frac{A+B}{2} \geq A\sharp B, \tag{1}$$

where $A\sharp B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ is called the geometric mean of A and B. For two Hermitian matrices A and B of the same size, in this article, we write $A \geq B$ (or $B \leq A$) to mean that $A - B$ is positive semidefinite.

If we denote $S := A\sharp B$, then $B = SA^{-1}S$. Thus a variant of (1) is the following

$$A, S \text{ are } n \times n \text{ positive definite matrices} \implies A + SA^{-1}S \geq 2S. \tag{2}$$

There is a long tradition in matrix analysis of comparing eigenvalues or singular values. To proceed, let us fix some notation. The j-th largest singular value of a complex matrix A is denoted by $\sigma_j(A)$. If all the eigenvalues of A are real, then we denote its j-th largest one by $\lambda_j(A)$. By Weyl's Monotonicity Theorem [2, p. 63], (1) readily implies

$$\lambda_j(A+B) \geq 2\lambda_j(A\sharp B), \qquad j = 1, \ldots, n.$$

As far as the eigenvalues or singular values are considered, there are other versions of "geometric mean". Bhatia and Kittaneh studied this kind of inequalities over a twenty year period [4–6]. Their elegant results include the following: If A, B are $n \times n$ positive semidefinite matrices, then

$$\lambda_j(A+B) \geq 2\sqrt{\lambda_j(AB)} = 2\sigma_j(A^{\frac{1}{2}}B^{\frac{1}{2}}); \tag{3}$$

$$\lambda_j(A+B) \geq 2\lambda_j(A^{\frac{1}{2}}B^{\frac{1}{2}}) \tag{4}$$

for $j = 1, \ldots, n$.

To complete the picture in (3)–(4), they asked whether it is true

$$\lambda_j(A+B) \geq 2\sqrt{\sigma_j(AB)}, \qquad j = 1, \ldots, n?$$

This question was recently answered in the affirmative by Drury in his very brilliant work [7]. The purpose of this expository article is to revisit Drury's solution. Hopefully, some of our arguments would shed new insights into the beautiful result, which is now a theorem.

Please cite this article in press as: M. Lin, On Drury's solution of Bhatia & Kittaneh's question, Linear Algebra Appl. (2016), http://dx.doi.org/10.1016/j.laa.2016.02.025