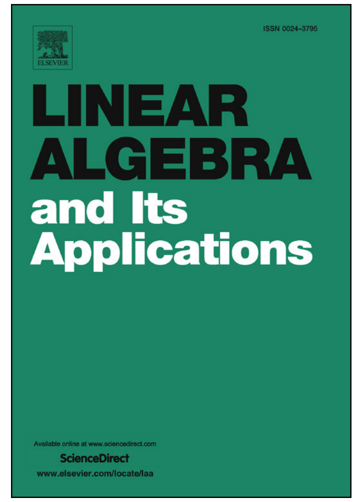

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A SYMMETRIZATION OF THE JORDAN CANONICAL FORM

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ABSTRACT. For a (finite or infinite dimensional) vector space V, the notion of a symmetric Jordan canonical form of an operator $T \in L(V)$ having a minimal polynomial is defined and used to verify the relation between the notions of "Jordan canonical form" and "rational canonical form." The paper extends and repairs Theorem 2.2 of [M. Radjabalipour, *The rational canonical form via the splitting field;* Linear Algebra and Its Applications, 439 (2013), no. 8, 2250-2255]. In particular, it is shown that there exists an auxiliary nilpotent operator $S \in L(W)$, depending on T, such that every Jordan canonical form of S yields a symmetric Jordan canonical form and, if the characteristic of the ground field is zero, a rational canonical form for T. The paper concludes with a direct proof of the symmetric Jordan canonical form which "integrates" into a rational canonical form.

Dedicated to Rajendra Bhatia

1. Introduction

Let T be a linear operator on a finite or infinite dimensional vector space V over a general field $\mathbb{F}$. For $v \in V$, let $Z(v, T)$ denote the cyclic invariant subspace of T spanned by $\{v, Tv, T^2v, \cdots\}$ (in the domain of T). The generator $f(x) := x^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0$ of the principal ideal domain consisting of all polynomials $g \in \mathbb{F}[x]$ satisfying $g(T)v = 0$ is called the minimal T-annihilator of v. The space $Z(v, T)$ is of finite dimension k if and only if the minimal T-annihilator of v has degree k. If $v_1, v_2, \cdots, v_k \in V$ have relatively prime minimal T-annihilators $g_1, g_2, \cdots, g_k$, then $v = v_1+v_2+\cdots+v_k$ has a minimal T-annihilator $h := h_1h_2\cdots h_k$ with $h_i|g_i\ \forall i$; since $(h_1g_2g_3\cdots g_k)(T)v_1 = 0$, it follows that $g_1|h_1$ and, thus, $g_1 = h_1$. By symmetry, $g_i = h_i\ \forall i$ and, in view of the primary decomposition theorem, there exist $w_i \in Z(v, T)$ such that $v = w_1 + w_2 + \cdots + w_k$ and w_i has a minimal T-annihilator $(i = 1, 2, \cdots, k)$ and

$$Z(v, T) = Z(w_1, T) \oplus Z(w_2, T) \oplus \cdots \oplus Z(w_k, T).$$

Now, $(g_2g_3\cdots g_k)(T)(v_1 - w_1) = 0$ which implies that $g_1|g_2g_3\cdots g_k$ if $v_1 \neq w_1$; the former being impossible, we conclude that $v_1 = w_1$ and, by symmetry,

$$Z(v, T) = Z(v_1, T) \oplus Z(v_2, T) \oplus \cdots \oplus Z(v_k, T). \tag{1.1}$$

The rational canonical form of T together with the corresponding cyclic decomposition of V assert that if V is finite dimensional [3, 4, 9] or, more generally, if T

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