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A note on inequalities for positive linear maps

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Dedicated to Professor Rajendra Bhatia on the occasion of his 65th birthday

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ABSTRACT

We obtain some generalisations of the inequalities for positive unital linear maps on the algebra of matrices. As a consequence, we obtain several positive semidefinite matrices and new eigenvalue inequalities for Hermitian matrices.

1. Introduction

Let $\mathbb{M}(n)$ be the C^*-algebra of all $n \times n$ complex matrices. Let $\Phi : \mathbb{M}(n) \to \mathbb{M}(l)$ be a positive unital linear map [3]. Kadison's inequality [11] says that for any Hermitian element A of $\mathbb{M}(n)$, we have

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$$\Phi(A^2) \geq \Phi(A)^2 \tag{1.1}$$

or equivalently

$$\begin{bmatrix} I & \Phi(A) \\ \Phi(A) & \Phi(A^2) \end{bmatrix} \geq O. \tag{1.2}$$

For more details, generalisations and extensions of this inequality, see Davis [9] and Choi [7,8].

A complementary inequality due to Bhatia and Davis [2] says that if the spectrum of a Hermitian matrix A is contained in the interval $[m, M]$, then

$$\Phi(A^2) - \Phi(A)^2 \leq \left(\frac{M-m}{2}\right)^2 I. \tag{1.3}$$

They also proved that

$$\Phi(A^2) - \Phi(A)^2 \leq (\Phi(A) - mI)(MI - \Phi(A)). \tag{1.4}$$

The inequality (1.4) provides a refinement of (1.3). For more details and applications of these inequalities, see [4–6].

The Kadison inequality (1.1) is a noncommutative version of the classical inequality

$$\mathbb{E}\left(X^2\right) \geq \mathbb{E}\left(X\right)^2, \tag{1.5}$$

where X is a random variable (a real-valued measurable function) on a probability space $(\Omega, \mathcal{F}, P)$ with finite expectation $\mathbb{E}(X) = \displaystyle\int X dP$. Kadison [11] remarks that the standard proofs of the corresponding inequalities for scalars do not apply to give simple proofs for linear maps. In case of Kadison's inequality (1.1), if $\Phi(A)$ and $\Phi(A^2)$ commute one can reduce the problem to the real-valued case, and the results follow from these considerations.

The inequality (1.5) is subsumed in a more general Jensen's inequality that says that if f is a convex function on (a, b) then

$$f(\mathbb{E}(X)) \leq \mathbb{E}(f(X)). \tag{1.6}$$

One generalisation of the Kadison inequality (1.1) is a noncommutative analogue of (1.6) that says that if A is a Hermitian matrix whose spectrum is contained in (a, b) and if f is matrix convex function [1] on (a, b) and Φ is a positive unital linear map, then

$$f(\Phi(A)) \leq \Phi(f(A)), \tag{1.7}$$

see Davis [9] and Choi [7]. Bhatia and Sharma [4] have shown that for 2×2 matrices the inequality (1.7) holds true for all ordinary convex functions on (a, b). It is well known that the same is true for every positive unital linear functional $\varphi : \mathbb{M}(n) \to \mathbb{C}$.

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