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The sum of squared logarithms inequality in arbitrary dimensions

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A B S T R A C T

We prove the *sum of squared logarithms inequality* (SSLI) which states that for nonnegative vectors $x, y \in \mathbb{R}^n$ whose elementary symmetric polynomials satisfy $e_k(x) \le e_k(y)$ (for $1 \le k < n$) and $e_n(x) = e_n(y)$, the inequality $\sum_i (\log x_i)^2 \le \sum_i (\log y_i)^2$ holds. Our proof of this inequality follows by a suitable extension to the complex plane. In particular, we show that the function $f\colon M \subseteq \mathbb{C}^n \to \mathbb{R}$ with $f(z) = \sum_i (\log z_i)^2$ has nonnegative partial derivatives with respect to the elementary symmetric polynomials of z. This property leads to our proof. We conclude by providing applications and wider connections of the SSLI.
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Algebraic geometry
Matrix analysis

1. Introduction

The *sum of squared logarithms inequality* (SSLI) arose first as a scientific issue in 2012 [22] while proving the optimality result:

$$\inf_{Q\in \mathrm{SO}(n)} \|\operatorname{sym}\operatorname{Log} Q^T F\|^2 = \inf_{Q\in \mathrm{SO}(n)} \inf_{\substack{Y\in\mathbb{R}^{n\times n}\\ \exp(Y)=Q^T F}} \|\operatorname{sym} Y\|^2 = \|\log\sqrt{F^T F}\|^2 , \qquad (1)$$

where $Y = \operatorname{Log} X$ denotes all solutions of the matrix exponential equation $\exp(Y) = X$, $\|\cdot\|$ denotes the Frobenius matrix norm, and $\operatorname{sym} X := \frac{1}{2}(X + X^T)$.

The SSLI (formally stated in Theorem 1.2) has been investigated in a series of works. In 2013, it was examined closely by Bîrsan, Neff, and Lankeit [7], who found a proof for $n \in \{2, 3\}$. For $n = 3$, the inequality can be written as follows: let $x_1, x_2, x_3, y_1, y_2, y_3 > 0$ be positive real numbers such that

$$\begin{aligned} x_1 + x_2 + x_3 &\le y_1 + y_2 + y_3 , \\ x_1\, x_2 + x_1\, x_3 + x_2\, x_3 &\le y_1\, y_2 + y_1\, y_3 + y_2\, y_3 , \\ x_1\, x_2\, x_3 &= y_1\, y_2\, y_3 . \end{aligned}$$

Then the sum of their squared logarithms satisfies the following inequality:

$$(\log x_1)^2 + (\log x_2)^2 + (\log x_3)^2 \le (\log y_1)^2 + (\log y_2)^2 + (\log y_3)^2 .$$

In 2015, Pompe and Neff [29] proved the SSLI for $n = 4$, based on a new idea that did not extend to higher dimensions without further complications. To state the SSLI for arbitrary n, we first recall

Definition 1.1. Let $x \in \mathbb{R}^n$. We denote by $e_k(x)$ the k-th *elementary symmetric polynomial*, i.e. the sum of all $\binom{n}{k}$ products of exactly k components of x:

$$e_k(x) := \sum_{1\le i_1<\ldots<i_k\le n} x_{i_1} x_{i_2} \ldots x_{i_k} \qquad \text{for any } k \in \{1, \ldots, n\} .$$

Note that $e_1(x) = x_1 + x_2 + \cdots + x_n$ and $e_n(x) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$.

We also write $\mathbb{R}_+ := \{x \in \mathbb{R} \,|\, x > 0\}$ and $\mathbb{R}_- := \{x \in \mathbb{R} \,|\, x < 0\}$ and set $\mathbb{R}^n_+ = (\mathbb{R}_+)^n$.

Theorem 1.2 *(Sum of squared logarithms inequality). Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^n_+$ such that*

$$e_k(x) \le e_k(y) \qquad \textit{for all } k \in \{1, \ldots, n-1\},$$

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