

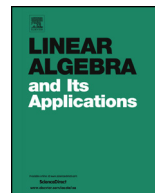


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# The sum of squared logarithms inequality in arbitrary dimensions

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## ABSTRACT

We prove the *sum of squared logarithms inequality* (SSLI) which states that for nonnegative vectors  $x, y \in \mathbb{R}^n$  whose elementary symmetric polynomials satisfy  $e_k(x) \leq e_k(y)$  (for  $1 \leq k < n$ ) and  $e_n(x) = e_n(y)$ , the inequality  $\sum_i (\log x_i)^2 \leq \sum_i (\log y_i)^2$  holds. Our proof of this inequality follows by a suitable extension to the complex plane. In particular, we show that the function  $f: M \subseteq \mathbb{C}^n \rightarrow \mathbb{R}$  with  $f(z) = \sum_i (\log z_i)^2$  has nonnegative partial derivatives with respect to the elementary symmetric polynomials of  $z$ . This property leads to our proof. We conclude by providing applications and wider connections of the SSLI.

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## 1. Introduction

The *sum of squared logarithms inequality* (SSLI) arose first as a scientific issue in 2012 [22] while proving the optimality result:

$$\inf_{Q \in \text{SO}(n)} \|\text{sym} \text{Log} Q^T F\|^2 = \inf_{Q \in \text{SO}(n)} \inf_{\substack{Y \in \mathbb{R}^{n \times n} \\ \exp(Y) = Q^T F}} \|\text{sym} Y\|^2 = \|\log \sqrt{F^T F}\|^2, \quad (1)$$

where  $Y = \text{Log} X$  denotes all solutions of the matrix exponential equation  $\exp(Y) = X$ ,  $\|\cdot\|$  denotes the Frobenius matrix norm, and  $\text{sym} X := \frac{1}{2}(X + X^T)$ .

The SSLI (formally stated in [Theorem 1.2](#)) has been investigated in a series of works. In 2013, it was examined closely by Birsan, Neff, and Lankeit [7], who found a proof for  $n \in \{2, 3\}$ . For  $n = 3$ , the inequality can be written as follows: let  $x_1, x_2, x_3, y_1, y_2, y_3 > 0$  be positive real numbers such that

$$\begin{aligned} x_1 + x_2 + x_3 &\leq y_1 + y_2 + y_3, \\ x_1 x_2 + x_1 x_3 + x_2 x_3 &\leq y_1 y_2 + y_1 y_3 + y_2 y_3, \\ x_1 x_2 x_3 &= y_1 y_2 y_3. \end{aligned}$$

Then the sum of their squared logarithms satisfies the following inequality:

$$(\log x_1)^2 + (\log x_2)^2 + (\log x_3)^2 \leq (\log y_1)^2 + (\log y_2)^2 + (\log y_3)^2.$$

In 2015, Pompe and Neff [29] proved the SSLI for  $n = 4$ , based on a new idea that did not extend to higher dimensions without further complications. To state the SSLI for arbitrary  $n$ , we first recall

**Definition 1.1.** Let  $x \in \mathbb{R}^n$ . We denote by  $e_k(x)$  the  $k$ -th *elementary symmetric polynomial*, i.e. the sum of all  $\binom{n}{k}$  products of exactly  $k$  components of  $x$ :

$$e_k(x) := \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k} \quad \text{for any } k \in \{1, \dots, n\}.$$

Note that  $e_1(x) = x_1 + x_2 + \dots + x_n$  and  $e_n(x) = x_1 \cdot x_2 \cdot \dots \cdot x_n$ .

We also write  $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$  and  $\mathbb{R}_- := \{x \in \mathbb{R} \mid x < 0\}$  and set  $\mathbb{R}_+^n = (\mathbb{R}_+)^n$ .

**Theorem 1.2** (*Sum of squared logarithms inequality*). Let  $n \in \mathbb{N}$  and  $x, y \in \mathbb{R}_+^n$  such that

$$e_k(x) \leq e_k(y) \quad \text{for all } k \in \{1, \dots, n-1\},$$

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