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Hadamard powers of some positive matrices

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A R T I C L E I N F O

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Positivity properties of the Hadamard powers of the matrix $[1 + x_i x_j]$ for distinct positive real numbers $x_1, \dots, x_n$ and the matrix $[|\cos((i-j)\pi/n)|]$ are studied. In particular, it is shown that the $n \times n$ matrix $[(1 + x_i x_j)^r]$ is positive semidefinite if and only if r is a nonnegative integer or $r > n - 2$, and for every odd integer $n \geq 3$ the $n \times n$ matrix $[|\cos((i-j)\pi/n)|^r]$ is positive semidefinite if and only if r is a nonnegative even integer or $r > n - 3$.

1. Introduction

Positive definite matrices are fundamental objects of study in matrix analysis and have applications in diverse areas such as engineering, statistics, quantum information, medical imaging and mechanics. A classical problem in matrix analysis involves the study of functions that act entrywise on matrices and preserve positivity. See, for instance, [6–10,13,14]. In particular, the study of entrywise power functions $t \mapsto t^r$ has been of special interest to various mathematicians; see [1,5,6,8].

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According to a theorem of Schur, the mth Hadamard power $A^{\circ m} = \left[a_{ij}^m\right]$ of a positive semidefinite matrix $A = [a_{ij}]$ is again positive semidefinite for every positive integer m. A positive semidefinite matrix is called *doubly nonnegative* if all its entries are nonnegative. If A is a doubly nonnegative matrix and r is a positive real number, then the rth *Hadamard power* of A is the matrix $A^{\circ r} = \left[a_{ij}^r\right]$.

FitzGerald and Horn [5] extensively studied the Hadamard powers of doubly nonnegative matrices. They showed that $n-2$ is the 'critical exponent' for $n \times n$ doubly nonnegative matrices, i.e., $n-2$ is the least number for which $A^{\circ r}$ is doubly nonnegative for every $n \times n$ doubly nonnegative matrix A and $r \geq n-2$. They also showed that for every positive real number $r < n-2$ that is not an integer, we can find a doubly nonnegative matrix whose rth Hadamard power is not positive semidefinite.

If A has arbitrary real (not necessarily nonnegative) entries, we consider a natural extension of real Hadamard powers. For a real positive semidefinite matrix A and a positive real number r, we denote the matrix $[|a_{ij}|^r]$ by $|A|_\circ^{\circ r}$. In particular if $r = 1$, then we denote the matrix $[|a_{ij}|]$ by $|A|_\circ$. If A is a 2×2 real positive semidefinite matrix, then $|A|_\circ^{\circ r}$ is positive semidefinite for all positive real r. But this is not true for higher dimensions. In fact, for every positive real r that is not an even integer, we can find a real positive semidefinite matrix A for which $|A|_\circ^{\circ r}$ is not positive semidefinite. (See [1].) When r is an even positive integer, then $a_{ij}^r = |a_{ij}|^r$. Hence by Schur's theorem, $|A|_\circ^{\circ r}$ is positive semidefinite in this case. Hiai [8] proved an analogue of the theorem of FitzGerald and Horn for $n \times n$ real positive semidefinite matrices. He showed that for every $n \times n$ real positive semidefinite matrix A, $n-2$ is the least number for which $|A|_\circ^{\circ r}$ is positive semidefinite for all $r \geq n-2$.

Recently there has been a renewed interest in the study of positivity properties of Hadamard powers of positive semidefinite matrices. Motivated by problems occurring in statistics, Guillot, Khare and Rajaratnam have been studying various problems related to Hadamard powers. See [6,7].

If $r < n-2$, there are two classes of examples in the literature. FitzGerald and Horn [5] considered the $n \times n$ doubly nonnegative matrix A with i,jth entry $(1 + \varepsilon ij)$ and showed that if r is not an integer and $0 < r < n-2$, then $A^{\circ r}$ is not positive semidefinite for a sufficiently small positive number ε. In this paper we show that this remains true if we replace εij with $x_i x_j$ for any distinct positive real numbers $x_1, \ldots, x_n$.

Theorem 1.1. *Let $x_1, \ldots, x_n$ be distinct positive real numbers. Let X be the $n \times n$ matrix*

$$X = [1 + x_i x_j]. \tag{1.1}$$

Then $X^{\circ r}$ is positive semidefinite if and only if r is a nonnegative integer or $r > n-2$. In particular, $X^{\circ r}$ is positive definite if $r > n-2$.

Bhatia and Elsner [1] studied another interesting class of $n \times n$ positive semidefinite Toeplitz matrices with real entries

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