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A class of simple matrix algebras ☆

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A R T I C L E I N F O

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Dedicated to Professor Rajendra Bhatia on the occasion of his 65th birthday

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A B S T R A C T

Fix an infinite set I and consider the associative matrix algebra $\mathcal{M}_I(\mathbb{F})$ where $\mathbb{F}$ is a base field with $\mathrm{char}(\mathbb{F}) \neq 2$. For any couple of bijective maps $\sigma, \nu : I \to I$, such that $\sigma\nu = \nu\sigma$ and $\sigma^2 = \nu^2$, we introduce a linear subspace $\Omega(\sigma, \nu)$ of $\mathcal{M}_I(\mathbb{F})$. We endow it with a structure of (non-associative) algebra for a certain bilinear product, and obtain a wide class of non-associative algebras containing, in particular, the Lie algebras $\mathrm{Lie}(\mathcal{M}_I(\mathbb{F}), t)$. We show that each algebra $\Omega(\sigma, \nu)$ is simple.

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1. Introduction and previous definitions

Fix a base field $\mathbb{F}$ with char($\mathbb{F}$) $\neq 2$, an infinite set of indexes I, and the linear space of $(I \times I)$-matrices over $\mathbb{F}$, denoted by $\mathcal{M}_I(\mathbb{F})$. That is, $\mathcal{M}_I(\mathbb{F})$ is the set of mappings $I \times I \to \mathbb{F}$ with just a finite number of non-zero values. For any $(i,j) \in I \times I$, we will denote by $E_{i,j}$ the elementary matrix given by $E_{i,j}(i,j) = 1$ and $E_{i,j}(p,q) = 0$ if $(p,q) \neq (i,j)$.

For any bijective map $\sigma : I \to I$ we will denote by

$$F_\sigma : \mathcal{M}_I(\mathbb{F}) \to \mathcal{M}_I(\mathbb{F})$$

the linear bijection determined by $F_\sigma(E_{i,j}) = E_{\sigma(i),\sigma(j)}$, which is an associative automorphism of $\mathcal{M}_I(\mathbb{F})$. Also, for any bijective map $\nu : I \to I$ we will denote by

$$G_\nu : \mathcal{M}_I(\mathbb{F}) \to \mathcal{M}_I(\mathbb{F})$$

the linear bijection defined by $G_\nu(E_{i,j}) = E_{\nu(j),\nu(i)}$, in which G_ν is an associative anti-automorphism of $\mathcal{M}_I(\mathbb{F})$. That is, $G_\nu(AB) = G_\nu(B)G_\nu(A)$ for any $A, B \in \mathcal{M}_I(\mathbb{F})$.

Fix now a couple of bijective maps $\sigma, \nu : I \to I$ and consider the associative automorphism F_σ and the associative anti-automorphism G_ν, constructed as above. Let us define the linear subspace of $\mathcal{M}_I(\mathbb{F})$ given by

$$\Omega(\sigma,\nu) := \{M \in \mathcal{M}_I(\mathbb{F}) : F_\sigma(M) = -G_\sigma(M)\}.$$

Example 1.1. If $\sigma = \nu = Id$, then F_σ is the identity on $\mathcal{M}_I(\mathbb{F})$ while G_ν is the transposition map. Hence, $\Omega(Id, Id) = \mathrm{Lie}(\mathcal{M}_I(\mathbb{F}), t)$.

We note that if $I = \mathbb{N}$, then we obtain a known countably dimensional simple orthogonal Lie algebra $\mathfrak{so}_\infty(\mathbb{F})$ which is a direct limit of finite dimensional simple orthogonal Lie algebras $\mathfrak{so}(n, \mathbb{F})$ under natural embeddings; see [2].

Example 1.2. Let us fix $I = \mathbb{C}$ the set of complex numbers, $\mathbb{F} = \mathbb{R}$ the field of real numbers, and consider the linear space $\mathcal{M}_\mathbb{C}(\mathbb{R})$. If we fix $\sigma, \nu : \mathbb{C} \to \mathbb{C}$ as $\sigma(z) = \bar{z}$, the conjugation map, and $\nu = Id$, we can consider the linear space

$$\Omega(\sigma, Id) = \{M \in \mathcal{M}_\mathbb{C}(\mathbb{R}) : F_\sigma(M) = -M^t\}.$$

Suppose now that $\sigma\nu = \nu\sigma$ and $\sigma^2 = \nu^2$. Then we can introduce the bilinear product

$$\langle \cdot, \cdot \rangle : \Omega(\sigma,\nu) \times \Omega(\sigma,\nu) \to \Omega(\sigma,\nu)$$

on the linear subspace $\Omega(\sigma,\nu)$ as follows:

$$\langle A, B \rangle := F_\sigma(AB) - G_\sigma(AB)$$

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