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Outer inverses: Characterization and applications

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ABSTRACT

We characterize the elements with outer inverse in a semigroup S, and provide explicit expressions for the class of outer inverses b of an element a such that $bS \subseteq yS$ and $Sb \subseteq Sx$, where x, y are any arbitrary elements of S. We apply this result to characterize pairs of outer inverses of given elements from an associative ring R, satisfying absorption laws extended for the outer inverses. We extend the result on right–left symmetry of $aR \oplus bR = (a+b)R$ (Jain–Prasad, 1998) to the general case of an associative ring. We conjecture that 'given an outer inverse x of a regular element a in a semigroup S, there exists a reflexive generalized inverse y of a such that $x \leq^{-} y$' and prove the conjecture when S is an associative ring.

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Absorption law
Associative ring

1. Preliminaries

In this article, we make use of simple but interesting characterizations of regular elements, and elements with outer inverse in a semigroup, to obtain explicit expressions for the class of outer inverses with the property like range inclusion in the context of matrices. Throughout this article, S denotes a semigroup and R denotes an associative ring. The semigroup S and associative ring R need not have multiplicative identities.

Definition 1. An element a in a semigroup S is said to be *regular* (or von Neumann regular) if there exists an element b in S satisfying the equation

$$aba = a, \tag{1}$$

in which case b is said to be a *generalized inverse* of a. If an element c satisfies the equation

$$cac = c, \tag{2}$$

then c is called an *outer inverse* of a. Further, b is said to be a *reflexive generalized inverse* of a if $a = aba$ and $b = bab$.

An arbitrary outer inverse of a is denoted by $a^{=}$, a generalized inverse of a by a^{-} and a reflexive generalized inverse of a by a_r^{-}. Readers are referred to [1–3] for definitions and properties of different types of outer inverses (pseudo inverses). We refer to [4] and [5] for basic notions in the theory of generalized inverse of matrices. Our discussion is confined to results associated with generalized inverses and outer inverses. They can be extended to other generalized inverses (e.g., Moore–Penrose inverse and core–EP generalized inverse).

The absorption law "$a^{-1}(a+b)b^{-1} = a^{-1} + b^{-1}$" has been extended to singular elements of an associative ring by several authors, e.g., [6,7]. They considered the problem of finding equivalent conditions for the absorption laws in terms of the Moore–Penrose, group, core inverse, core inverse dual, and $\{1\}$, $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$ inverses in rings with identity. In [6,7], the authors characterized the elements a, b satisfying $a^{-}(a+b)b^{-} = a^{-}+b^{-}$ for all a^{-} and b^{-} belonging to a certain class of generalized inverse. In Section 3, we readdress the absorption law by considering $a^{=}(a+b)b^{=} = a^{=}+b^{=}$ for some $a^{=}$ and $b^{=}$. This modification is inspired by the observation that $e_1(e_1+e_2)e_2 = e_1+e_2 \Leftrightarrow e_1 = e_2$, where e_1 and e_2 are any real or complex idempotent matrices. We characterize every pair of outer inverses $a^{=}$ and $b^{=}$ that satisfy the modified absorption law.

An *idempotent* element e in S is a regular element in S, as $e^2 = e$. We say that two elements a, b of a semigroup are *space equivalent* if

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