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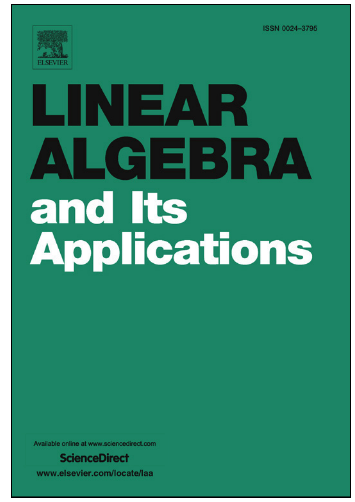
Dinh Trung Hoa



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SOME INEQUALITIES FOR THE MATRIX HERON MEAN

DINH TRUNG HOA

Abstract. In this paper, we prove some norm inequalities for the matrix Heron mean of two positive definite matrices. We also prove a determinant inequality for the t-power mean of two positive definite matrices. As a consequence, we obtain a determinant inequality for the matrix Heron mean.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices and M_n^+ the cone of positive semidefinite matrices. Denote by I the identity element of M_n. For self-adjoint matrices $A, B \in M_n$ the notation $A \le B$ means that $B - A \in M_n^+$. For a real-valued function f of a real variable and a self-adjoint matrix $A \in M_n$, the value $f(A)$ is understood by means of the functional calculus.

For $0 \le t \le 1$ the t-weighted geometric mean of A and B is defined as

$$A\sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}.$$

The geometric mean $A\sharp B := A\sharp_{1/2}B$ is the midpoint of the unique geodesic $A\sharp_t B$ connecting two points A and B in the Riemannian manifold of positive definite matrices with trace metric $ds = ||A^{-1/2}dA\ A^{-1/2}||_F = (\mathrm{Tr}(A^{-1}dA)^2)^{1/2}$ (cf. [7] or [8]).

Recently, Bhatia et al. [1] proved that for any positive definite matrices A and B and for $p = 1, 2$,

$$||A + B + 2rA\sharp B||_p \le ||A + B + r(A^{1/2}B^{1/2} + B^{1/2}A^{1/2})||_p. \tag{1}$$

When $r = 1$, inequality holds for $p = \infty$.

For the case $p = 2$ the proof of (1) is based on the following fact: for any positive definite matrices A and B,

$$\lambda(A^{1/2}(A\sharp B)A^{1/2}) \prec_{\log} \lambda(A^{3/4}B^{1/2}A^{3/4}), \tag{2}$$

where the notation λ is used for the n-tuple of eigenvalues of a matrix A in decent order and $\lambda(A) \prec_{\log} \lambda(B)$ means that

$$\prod_{j=1}^{k} \lambda_i(A) \le \prod_{j=1}^{k} \lambda_i(B), \quad 1 \le k \le n$$

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