ARTICLE IN PRESS LAA:14102

Linear Algebra and its Applications ••• (••••) •••–•••

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Unitaries permuting two orthogonal projections

Barry Simon [1]

Departments of Mathematics and Physics, Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125, United States

ARTICLE INFO

Article history:
Received 15 March 2017
Accepted 27 March 2017
Available online xxxx
Submitted by P. Semrl

MSC:
47A05
47A46
47A53

Keywords:
Pairs of projections
Index

ABSTRACT

Let P and Q be two orthogonal projections on a separable Hilbert space, $\mathcal{H}$. Wang, Du and Dou proved that there exists a unitary, U, with $UPU^{-1} = Q$, $UQU^{-1} = P$ if and only if $\dim(\ker P \cap \ker(1 - Q)) = \dim(\ker Q \cap \ker(1 - P))$ (both may be infinite). We provide a new proof using the supersymmetric machinery of Avron, Seiler and Simon.

I am delighted at this opportunity to present a birthday bouquet to Rajendra Bhatia whom I have long admired. He once told me that he had learned functional analysis from Reed–Simon. He more than returned the favor since I've learned so much from his books especially that much of matrix theory is actually analysis. In particular, my interest in Loewner's theorem on monotone matrix functions was stirred by his clear presentation of the Krein–Millman proof of that result. As I've been writing my own monograph on Loewner's Theorem, I discovered several time areas of application and extension of that

E-mail address: bsimon@caltech.edu.

[1] Research supported in part by NSF grant DMS-1265592 and in part by Israeli BSF Grant No. 2010348.

http://dx.doi.org/10.1016/j.laa.2017.03.026
0024-3795/© 2017 Published by Elsevier Inc.

Please cite this article in press as: B. Simon, Unitaries permuting two orthogonal projections, Linear Algebra Appl. (2017), http://dx.doi.org/10.1016/j.laa.2017.03.026

result where Bhatia was a key figure and where invariably his lucid prose helped me in absorbing the developments.

Let P and Q be two orthogonal projections on a separable Hilbert space, $\mathcal{H}$. It is a basic result in eigenvalue perturbations theory that when

$$\|P - Q\| < 1 \tag{1}$$

there exists a unitary U so that

$$UP = QU \tag{2}$$

It is even known that there exist unitaries, U, so that

$$UPU^{-1} = Q, \quad UQU^{-1} = P \tag{3}$$

The simpler question involving (2) goes back to Sz.-Nagy [14] and was further studied by Kato [10] who found a cleaner formula for U than Sz.-Nagy, namely Kato used

$$U = [QP + (1 - Q)(1 - P)] \left[1 - (P - Q)^2\right]^{-1/2} \tag{4}$$

Using Nagy's formula, Wolf [16] had extended this to arbitrary pairs of projections on a Banach space (requiring only that U is invertible rather than unitary) so long as

$$\|P - Q\|\|P\|^2 < 1 \qquad \|P - Q\|\|Q\|^2 < 1 \tag{5}$$

For non-orthogonal projections and projections on a Banach space, in general, $\|P\| \geq 1$ with equality in the Hilbert space case only if P is orthogonal so (5) is strictly stronger than (1). One advantage of Kato's form (4), is that in the Banach space case where the square root can be defined by a power series, it only requires (1).

For the applications they had in mind, it is critical not only that U exists but that on the set of pairs that (1) holds, U is analytic in P and Q. For they considered an analytic family, $A(z)$, and λ_0 an isolated eigenvalue of $A(0)$ of finite algebraic multiplicity. Then one can define

$$P(z) = \frac{1}{2\pi i} \oint\limits_{|\lambda - \lambda_0| = r} (\lambda - A(z))^{-1} d\lambda$$

for fixed small r and $|z|$ small. For $|z|$ very small, $\|P(z) - P(0)\| < 1$. If $U(z)$ is given by (4) with $Q = P(z)$, then $U(z)A(z)U(z)^{-1}$ leaves $\operatorname{ran} P(0)$ invariant and the study of eigenvalues of $A(z)$ near λ_0 is reduced to the finite dimensional problem $U(z)A(z)U(z)^{-1} \restriction \operatorname{ran} P(0)$. See the books of Kato [11], Baumgärtel [3] or Simon [13] for this subject.

Please cite this article in press as: B. Simon, Unitaries permuting two orthogonal projections, Linear Algebra Appl. (2017), http://dx.doi.org/10.1016/j.laa.2017.03.026