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Spectral sets and functions on Euclidean Jordan algebras

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ABSTRACT

Spectral sets (functions) in Euclidean Jordan algebras are generalizations of permutation invariant sets (respectively, functions) in $\mathcal{R}^n$. In this article, we study properties of such sets and functions and show how they are related to algebra automorphisms and majorization. We show that spectral sets/functions are indeed invariant under automorphisms, but the converse may not hold unless the algebra is $\mathcal{R}^n$ or simple. We study Schur-convex spectral functions and provide some applications. We also discuss the transfer principle and a related metaformula.

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1. Introduction

This paper deals with some interconnections between spectral sets/functions, algebra automorphisms, and majorization in the setting of Euclidean Jordan algebras. Given a Euclidean Jordan algebra $\mathcal{V}$ of rank n, a set E in $\mathcal{V}$ is said to be a *spectral set* [1] if it is of the form

$$E = \lambda^{-1}(Q),$$

where Q is a permutation invariant set in $\mathcal{R}^n$ and $\lambda : \mathcal{V} \to \mathcal{R}^n$ is the eigenvalue map (that takes x to $\lambda(x)$, the vector of eigenvalues of x with entries written in the decreasing order). A function $F : \mathcal{V} \to \mathcal{R}$ is said to be a *spectral function* [1] if it is of the form

$$F = f \circ \lambda,$$

where $f : \mathcal{R}^n \to \mathcal{R}$ is a permutation invariant function. How these are related to algebra automorphisms of $\mathcal{V}$ (which are invertible linear transformations on $\mathcal{V}$ that preserve the Jordan product) and majorization is the main focus of the paper.

The above concepts are generalizations of similar concepts that have been extensively studied in the setting of $\mathcal{R}^n$ (where the concepts reduce to permutation invariant sets and functions) and in $\mathcal{S}^n$ ($\mathcal{H}^n$), the space of all $n \times n$ real (respectively, complex) Hermitian matrices, see for example, [3–5,10–14,20], and the references therein. In the case of $\mathcal{S}^n$ ($\mathcal{H}^n$), spectral sets/functions are precisely those that are invariant under linear transformations of the form $X \to UXU^*$, where U is an orthogonal (respectively, unitary) matrix.

There are a few works that deal with spectral sets and functions on general Euclidean Jordan algebras. Baes [1] discusses some properties of Q that get transferred to E (such as closedness, openness, boundedness/compactness, and convexity) and properties of f that get transferred to F (such as convexity and differentiability). Sun and Sun [22] deal with the transferability of the semismoothness properties of f to F. Ramirez, Seeger, and Sossa [18] and Sossa [21] deal with a commutation principle and a number of applications.

In this paper, we present some new results on spectral sets and functions. In addition to giving some elementary characterization results, we show that spectral sets/functions are indeed invariant under automorphisms of $\mathcal{V}$, but the converse may not hold unless the algebra is $\mathcal{R}^n$ or simple (e.g., $\mathcal{S}^n$ or $\mathcal{H}^n$). We will also relate the concepts of spectral sets and functions to that of majorization. Given two elements x, y in $\mathcal{V}$, we say that x *is majorized by* y and write $x \prec y$ if $\lambda(x)$ is majorized by $\lambda(y)$ in $\mathcal{R}^n$ (see Section 2.1 for the definition); we say that x and y are *spectrally equivalent* and write $x \sim y$ if $\lambda(x) = \lambda(y)$ (or equivalently, $x \prec y$ and $y \prec x$). We show that spectral sets are characterized by the condition

$$x \sim y,\, y \in E \quad \Rightarrow \quad x \in E$$