Linear Algebra and its Applications 518 (2017) 57–78

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Nilpotent linear spaces and Albert's Problem

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A R T I C L E   I N F O

Article history:
Received 7 July 2016
Accepted 20 December 2016
Available online 23 December 2016
Submitted by P. Semrl

MSC:
17A05
17A30
15A04

Keywords:
Nilpotent linear space
Nilalgebra
Albert's Problem
Power-associative

A B S T R A C T

In [8] M.A. Fasoli classified, up to conjugation, all the maximal vector subspaces of $M_4(\mathbb{C})$, in which all the elements are nilpotent matrices. This result will allow us to solve Albert's Problem [5] for commutative power-associative $\mathbb{C}$-nilalgebras of dimension n and nilindex $n-3$ in an affirmative way.

1. Preliminaries

All the algebras considered in this paper are commutative finite-dimensional and not necessarily associative over a field $\mathbb{F}$. The powers of an element of an algebra $\mathcal{A}$ are not uniquely defined. The *principal powers* of an element a of the algebra $\mathcal{A}$ are defined inductively by $a^1 = a$ and $a^{k+1} = aa^k$ for all $k \geq 1$. This element a is *(principal) nilpotent of index* $\leq t$ if $a^t = 0$. We say that the algebra $\mathcal{A}$ is a *(principal) nilalgebra* if

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http://dx.doi.org/10.1016/j.laa.2016.12.026
0024-3795/

every element a of $\mathcal{A}$ is nilpotent. The smallest positive integer t such that every element of the algebra is nilpotent of index $\leq t$ is called the *(principal) nilpotent nilindex* of the algebra.

A *nilpotent algebra* is an algebra for which there is a natural number t such that any product containing at least t elements of the algebra is zero. Thus, an algebra $\mathcal{A}$ is nilpotent of index $\leq t$ if $\mathcal{A}^t = 0$, where the powers of the algebra $\mathcal{A}$ are defined as follows: $\mathcal{A}^1 = \mathcal{A}$ and $\mathcal{A}^k = \sum_{i+j=k} \mathcal{A}^i\mathcal{A}^j$ for all $k \geq 2$. The algebra $\mathcal{A}$ is *solvable* of index less than or equal to t if the tth plenary power of $\mathcal{A}$ vanishes, $\mathcal{A}^{(t)} = 0$, where the plenary powers of $\mathcal{A}$ are defined inductively as follows: $A^{(0)} = \mathcal{A}$ and $\mathcal{A}^{(k)} = \mathcal{A}^{(k-1)}\mathcal{A}^{(k-1)}$ for all positive integers k.

A commutative algebra $\mathcal{A}$ is called a *power-associative* algebra if for any $a \in \mathcal{A}$, the subalgebra $\mathbb{F}[a]$ of $\mathcal{A}$ generated by a is associative. Commutative power-associative algebras are a natural generalization of associative, alternative, and Jordan algebras. It is well known that every finite-dimensional Jordan nilalgebra is nilpotent. In [1] A.A. Albert writes the following "*In the general power-associative ring case no such result is to be expected and indeed every simple Lie algebra is a nilring. One can then hardly expect to be able to prove that a nilring is nilpotent but a limited result of this type is provable.*" Gerstenhaber and Myung [16] proved that every commutative power-associative nilalgebra of dimension ≤ 4 and characteristic $\neq 2$ is nilpotent. In [18], D. Suttles gives an example of a commutative power-associative nilalgebra of dimension 5 which is not nilpotent but solvable. Thus, a modified version of Albert's problem was formulated in [5], Problem 1.1.

Albert's Problem. Is every finite-dimensional (commutative) power-associative nilalgebra solvable?

The problem is still open. In some particular cases, a positive answer for Albert's question was obtained. It has been proved [3,2,7,11,9] that commutative power-associative nilalgebras of dimension ≤ 8 and characteristic different from 2, 3 and 5 are solvable. For nilalgebras with large nilindex with respect to their dimension we know the following result [4,6,9].

Theorem 1. *Let $\mathcal{A}$ be a commutative power-associative nilalgebra with dimension n and nilindex $t \geq n - 2$, over a field of characteristic 0. Then, $\mathcal{A}$ is solvable.*

In a series of papers, M. Gerstenhaber [12–15] established connections between nilalgebras and vector spaces of nilpotent linear transformations. A generalization of Theorem 1 of [13] was obtained in [6].

Theorem 2. *Let $\mathcal{A}$ be an arbitrary commutative nilalgebra of bounded nilindex t over a field $\mathbb{F}$. If the characteristic of $\mathbb{F}$ is either zero or greater than $2t - 3$, then $L_a^{2t-3} = 0$ for all $a \in \mathcal{A}$. If either $t \leq 6$, $\mathcal{A}$ is power-associative or $\mathbb{F}$ has at least $2t - 3$ elements, then the result is also valid when the characteristic of $\mathbb{F}$ is at least t.*