Linear Algebra and its Applications 518 (2017) 79–96

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Special and exceptional mock-Lie algebras

Pasha Zusmanovich

Department of Mathematics, University of Ostrava, Ostrava, Czech Republic

ARTICLE INFO

Article history:
Received 29 August 2016
Accepted 21 December 2016
Available online 28 December 2016
Submitted by P. Semrl

MSC:
17C10
17C40
17C55
17B35
17-08

Keywords:
Mock-Lie algebra
Jordan algebra
Jacobi identity
Nil index 3
Faithful representation
Albert

ABSTRACT

We observe several facts and make conjectures about commutative algebras satisfying the Jacobi identity. The central question is which of those algebras admit a faithful representation (i.e., in Lie parlance, satisfy the Ado theorem, or, in Jordan parlance, are special).

0. Introduction

A while ago, a new class of algebras emerged in the literature – the so-called mock-Lie algebras. These are commutative algebras satisfying the Jacobi identity. These algebras are locally nilpotent, so there are no nontrivial simple objects. Nevertheless, they seem to have an interesting structure theory which gives rise to interesting questions. And,

E-mail address: pasha.zusmanovich@osu.cz.

http://dx.doi.org/10.1016/j.laa.2016.12.029

after all, it is always curios to play with a classical notion by modifying it here and there and see what will happen – in this case, to replace in Lie algebras anti-commutativity by commutativity.

In [1], it was asked whether a finite-dimensional mock-Lie algebra admits a finite-dimensional faithful representation. In fact, an example providing a negative answer to this question was given a while ago – hidden in a somewhat obscure place, [8]. We provide an independent computer verification of that and similar examples, and examine several constructions and arguments for Lie algebras to see what works and what breaks in the mock-Lie case.

1. Definitions. Preliminary facts and observations

The standing assumption throughout the paper is that the base field K is of characteristic $\neq 2, 3$. An algebra L over K, with multiplication denoted by $\circ$, is called *mock-Lie* if it is commutative:

$$x \circ y = y \circ x,$$

and satisfies the Jacobi identity:

$$(x \circ y) \circ z + (z \circ x) \circ y + (y \circ z) \circ x = 0$$

for any $x, y, z \in L$.

A substitution $x = y = z$ into the Jacobi identity yields $x^{\circ 3} = (x \circ x) \circ x = 0$. Conversely, linearizing the latter identity, we get back the Jacobi identity. Moreover, it is easy to see that assuming commutativity, the Jacobi identity is equivalent to the Jordan identity $(x^{\circ 2} \circ y) \circ x = x^{\circ 2} \circ (y \circ x)$ (see, for example, [3, Lemma 2.2]). (On the other hand, commutative Leibniz and commutative Zinbiel algebras form a narrower class of mock-Lie algebras, namely, commutative algebras of nilpotency index 3: $(x \circ y) \circ z = 0$.)

Thus, mock-Lie algebras can be characterized at least in the following four equivalent ways:

$$\frac{\text{commutative}}{\text{Jordan}} \text{ algebras } \frac{\text{satisfying the Jacobi identity}}{\text{of nil index 3}}.$$

This class of algebras appeared in the literature under different names, reflecting, perhaps, the fact that it was considered from different viewpoints by different communities, sometimes not aware of each other's results. Apparently, for the first time these algebras appeared in [22, §5], where an example of infinite-dimensional solvable but not nilpotent mock-Lie algebra was given (reproduced in [23, §4.1, Example 1]); further examples can be found in [23, §4.1, Example 2 and §5.4, Exercise 4] and [20, §2.5]. In this and other Jordan-algebraic literature these algebras are called just "Jordan algebras