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On best uniform approximation by low-rank matrices

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A B S T R A C T

We study the problem of best approximation, in the element-wise maximum norm, of a given matrix by another matrix of lower rank. We generalize a recent result by Pinkus that describes the best approximation error in a class of low-rank approximation problems and give an elementary proof for it. Based on this result, we describe the best approximation error and the error matrix in the case of approximation by a matrix of rank one less than the original one. For the case of approximation by matrices with arbitrary rank, we give lower and upper bounds for the best approximation error in terms of certain submatrices of maximal volume. We illustrate our results using 2×2 matrices as examples, for which we also give a simple closed form of the best approximation error.

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1. Introduction

We consider the problem of approximating a given matrix as closely as possible by a matrix of the same size, but lower rank. When measuring the approximation error in the spectral or Frobenius norms, a full description of the best approximation and its error is given in terms of the singular value decomposition [10,3]. In different matrix norms, very little was known about this approximation problem until a recent article by Pinkus [8], where approximation by a class of elementwise norms, and there in particular ℓ_1-like norms, was studied. Pinkus derives expressions for the best approximation error in such norms and, in particular cases, shows that a best approximating matrix matches the original matrix in a number of rows and columns.

In the present paper, we derive analogues of several of Pinkus' results for the case of approximation in the elementwise maximum norm. In the process, we generalize one core result from [8] and prove it using only known basic results on best approximation, whereas the proof of the original result relied heavily on the theory of n-widths. Building on this result, we obtain an expression for the best approximation error of a matrix by another one with rank one less, as well as a characterization of the matrix of best approximation.

For approximation where the difference in ranks is greater than one, we have no closed formula for the best approximation error, but give lower and upper bounds for it involving certain submatrices of maximal volumes, that is, with greatest modulus of their determinants. These results are similar to some given by Babaev [1] in the continuous setting. The relevance of submatrices of maximal volume to the problem of low-rank approximation was first established by Goreinov and Tyrtyshnikov [4].

The remainder of the paper is structured as follows. In Section 2, we state the low-rank approximation problem and prove a result on the best approximation error which generalizes a result by Pinkus. In Section 3, we focus on the case of approximating a matrix by another matrix of rank one less, where the best approximation error can be described quite closely and we obtain an equioscillation result for the error matrix. In Section 4, we deal with approximations of arbitrary rank and give lower and upper bounds for the best approximation error in terms of certain submatrices of maximal volume. Finally, in Section 5, we illustrate some of our results in the simple case of 2×2-matrices, where we are also able to give a simple closed form for the best approximation error.

2. Approximation with low-rank matrices

2.1. Problem statement

Let $A \in \mathbb{R}^{m\times n}$ and $p, q \in [1, \infty]$. We define the entrywise matrix norm

$$|A|_{p,q} := \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} |a_{ij}|^q \right)^{p/q} \right)^{1/p},$$