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Weighted least squares solutions of the equation $AXB - C = 0$

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ABSTRACT

Let $\mathcal{H}$ be a Hilbert space, $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$ and $W \in L(\mathcal{H})$ a positive operator such that $W^{1/2}$ is in the p-Schatten class, for some $1 \leq p < \infty$. Given $A, B \in L(\mathcal{H})$ with closed range and $C \in L(\mathcal{H})$, we study the following weighted approximation problem: analyze the existence of

$$\underset{X \in L(\mathcal{H})}{min} \|AXB - C\|_{p,W}, \tag{0.1}$$

where $\|X\|_{p,W} = \|W^{1/2}X\|_p$. We also study the related operator approximation problem: analyze the existence of

$$\underset{X \in L(\mathcal{H})}{min} (AXB - C)^*W(AXB - C), \tag{0.2}$$

where the order is the one induced in $L(\mathcal{H})$ by the cone of positive operators. In this paper we prove that the existence of the minimum of (0.2) is equivalent to the existence of a solution of the normal equation $A^*W(AXB - C) = 0$. We also give sufficient conditions for the existence of the minimum

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of (0.1) and we characterize the operators where the minimum is attained.

1. Introduction

In signal processing language, *sampling* is an operation which converts a continuous signal (modelled as a vector in an adequate Hilbert space $\mathcal{H}$) into a discrete one.

Frequently the samples of a signal $f \in \mathcal{H}$ are represented in the following way: given a frame $\{v_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ of a closed subspace $\mathcal{S}$, called the sampling subspace, the samples are given by $\{f_n\}_{n\in\mathbb{N}} = \{\langle f, v_n \rangle\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$. On the other hand, given samples $\{f_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, the reconstructed signal $\hat{f}$ is given by $\hat{f} = \sum_{n\in\mathbb{N}} f_n w_n$, where $\{w_n\}_{n\in\mathbb{N}}$ is a frame of the closed subspace $\mathcal{R}$, called the reconstruction subspace.

Suppose that, A and B are the *synthesis operators* corresponding to the frames $\{w_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$, respectively, i.e. $A, B : \ell^2(\mathbb{N}) \to \mathcal{H}$ are the operators such that, if $x = \{x_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, $Ax = \sum_{n\in\mathbb{N}} x_n w_n$ and $Bx = \sum_{n\in\mathbb{N}} x_n v_n$, which are bounded since $\{v_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are frames. Observe that, the samples of f are given by $\{f_n\}_{n\in\mathbb{N}} = B^*f$ and, given samples $\{f_n\}_{n\in\mathbb{N}}$ the reconstructed signal is given by $\hat{f} = A(\{f_n\}_{n\in\mathbb{N}})$, see [13,25].

If we only know the samples of a signal $\{f_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, in general it is not possible to recover the signal $f \in \mathcal{H}$, even if we apply a digital filter (a bounded linear operator $X : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$) to these samples. But, in some cases it is possible to find a good representation of the signal $f \in \mathcal{H}$, i.e., a recovered signal $\hat{f} = AXB^*f$ that has good properties. For instance, in the classical sampling scheme (where sampling and reconstruction subspaces coincide) it is possible to reconstruct the best approximation of the signal f, i.e., it is possible to find X such that $AXB^* = P_{\mathcal{S}}$ and then $\hat{f} = P_{\mathcal{S}}f$, where $P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S} = \mathcal{R} = R(A)$. Another interesting example, where the sampling and reconstruction subspaces may not coincide, is the so called consistent sampling scheme, where the samples of the reconstructed signal $\hat{f}$ are equal to the samples of the original signal f, i.e. $B^*\hat{f} = B^*f$, in this case X is such that $Q = AXB^*$ turns out to be an oblique projection. Consequently, the reconstructed signal $\hat{f}$ is not necessarily a good approximation of f, since the distance $\|f - \hat{f}\| = \|f - AXB^*f\|$ is not minimized. Now suppose we want to find a digital filter $X \in \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ such that AXB^*f is a good approximation of f in $R(A) = \mathcal{R}$, i.e. we want that AXB^* approximates $P_{\mathcal{R}}$ in some sense. For instance, we may want to find $X_0 : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ a bounded linear operator, such that, for every $f \in \mathcal{H}$

$$\|(AX_0B^* - P_{\mathcal{R}})f\| \leq \|(AXB^* - P_{\mathcal{R}})f\|,$$

for every $X \in L(\mathcal{H})$ (the algebra of linear bounded operators on $\mathcal{H}$). This means that we are interested in the problem: