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Existence of a not necessarily symmetric matrix with given distinct eigenvalues and graph

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For given distinct numbers $\lambda_1 \pm \mu_1 \mathrm{i}, \lambda_2 \pm \mu_2 \mathrm{i}, \ldots, \lambda_k \pm \mu_k \mathrm{i} \in \mathbb{C} \setminus \mathbb{R}$ and $\gamma_1, \gamma_2, \ldots, \gamma_l \in \mathbb{R}$, and a given graph G with a matching of size at least k, we will show that there is a real matrix whose eigenvalues are the given numbers and its graph is G. In particular, this implies that any real matrix with distinct eigenvalues is similar to a real, irreducible, tridiagonal matrix.

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1. Introduction

A *directed graph* $G = (V, E)$ is a pair of sets V and E where V is the set of vertices of G, and E, the set of edges of G, is a subset of $V \times V$. That is, each element of E is an ordered pair (u, v), with $u, v \in V$. We say a graph G is *loopless* when for each $(u, v) \in E$, we have $u \neq v$. In this paper we only consider loopless graphs. If $(u, v) \in E$ then we say u is adjacent to v and denote it by $u \to v$. Note that such a graph might have both edges (u, v) and (v, u), but since E is a set, there are no multiple edges from u to v. A directed loopless graph $G = (V, E)$ is said to have a *matching* of size k if E contains k vertex-disjoint edges $(u_1, v_1), \ldots, (u_k, v_k)$ and their reverses $(v_1, u_1), \ldots, (v_k, u_k)$.

A graph G is called undirected if for each $u \neq v$ the edge $(u, v) \in E$ if and only if $(v, u) \in E$. Hence we can ignore the directions of edges and consider E as a set of 2-subsets of V. That is, $E \subset \big\{\{u, v\} \,\big|\, u, v \in V\big\}$. In this case we call G an *undirected graph*. An undirected loopless graph $G = (V, E)$ is said to have a *matching* of size k if E contains k vertex-disjoint edges $\{u_1, v_1\}, \ldots, \{u_k, v_k\}$

Let $A \in \mathbb{R}^{n \times n}$. We say a (directed or undirected) loopless graph G is the *graph of the matrix* A when for each $i \neq j$ we have $A_{i,j} \neq 0$ if and only if $i \to j$. Note that the diagonal entries of A can be zero or nonzero.

It is of interest to study the existence of matrices with given spectral properties and graph, see [1, Chapter 4]. For the problems when the solution matrix is not necessarily symmetric see [2] for a survey on the structured inverse eigenvalue problems with an extensive bibliography, specially SIEP6b, and see [3] for the related minimum rank problems. In this paper we examine a the problem of existence of a solution matrix $A \in \mathbb{R}^{n \times n}$ where G, the graph of A, and Λ, its spectrum, are prescribed. An obvious necessary condition for the existence of a solution is that Λ to be closed under complex conjugation. Assume that Λ consists of k distinct complex conjugate pairs in $\mathbb{C} \setminus \mathbb{R}$. We will show that a sufficient condition for the existence of a solution is that

C1. G has a matching of size at least k, and
C2. all the eigenvalues are distinct.

Example 1.1 shows that condition C1 is not necessary, and Example 1.2 shows that condition C2 is not necessary.

Example 1.1. Let G be a star on 5 vertices with 1 as the center vertex, and let

$$A = \begin{bmatrix} 3 & -2 & -6 & 1 & 6 \\ 5 & -2 & 0 & 0 & 0 \\ -5 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 3 & 0 \\ -6 & 0 & 0 & 0 & 5 \end{bmatrix},$$

G: