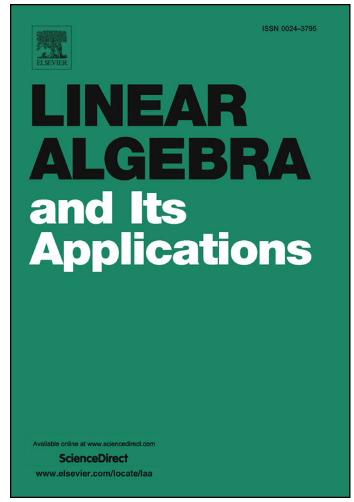

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COMPLEMENTARY EIGENVALUES OF GRAPHS

RAFAEL FERNANDES, JOAQUIM JUDICE, AND VILMAR TREVISAN

Abstract. In this paper, we study the Eigenvalue Complementarity Problem (EiCP) when its matrix A belongs to the class $S(G) = \{A = [a_{ij}] : a_{ij} = a_{ji} \neq 0 \text{ iff } ij \in E\}$, where $G = (V, E)$ is a connected graph. It is shown that if all non diagonal elements of $A \in S(G)$ are non positive, then A has a unique complementary eigenvalue, which is the smallest eigenvalue of A. In particular, zero is the unique complementary eigenvalue of the Laplacian and the normalized Laplacian matrices of a connected graph. The number $\mathfrak{c}(G)$ of complementary eigenvalues of the adjacency matrix of a connected graph G is shown to be bounded above by the number $\mathfrak{b}(G)$ of induced non isomorphic connected subgraphs of G. Furthermore, $\mathfrak{c}(G) = \mathfrak{b}(G)$ if the Perron roots of the adjacency matrices of these subgraphs are all distinct. Finally, the maximum number of complementary eigenvalues for the adjacency matrices of graphs is shown to grow faster than any polynomial on the number of vertices.

Keywords: Eigenvalue Problem; Complementarity Problem; Graphs; Spectral Graph Theory.
MSC: 05C50

1. Introduction

Given a real matrix A of order n, the Eigenvalue Complementarity Problem (EiCP) consists of finding a real number λ and a vector $x \in \mathbb{R}^n - \{0\}$ such that

$$w = Ax - \lambda x \tag{1}$$

$$x \geq 0, \ \ w \geq 0 \tag{2}$$

$$x^T w = 0 \tag{3}$$

where $w \in \mathbb{R}^n$. Each solution (λ, x) of EiCP satisfies the feasibility conditions (1) and (2) and the condition (3). Since x and w are nonnegative vectors then this last condition is equivalent to n conditions:

$$x_i w_i = 0, \ \ i = 1, \ldots, n \tag{4}$$

So, for each i at most one of the variables x_i or w_i may be positive. These variables are called complementary [7, 9] and the constraint (3) is named the complementarity condition. If $w = 0$ and x is not required to be nonnegative then EiCP reduces to the well-known Eigenvalue Problem (EiP) [14]:

$$Ax = \lambda x \tag{5}$$

Therefore, EiCP is an extension of EiP that contains a complementarity condition (3) on nonnegative variables. In each solution (λ, x) of EiCP, λ is called a complementary eigenvalue and x is an associated complementary eigenvector. Furthermore EiCP (1) — (3) is said to be symmetric if its matrix A is symmetric. The words Pareto eigenvalue and Pareto eigenvector have been used by many authors to name these complementary eigenvalue and eigenvector (see for instance [31]).

EiCP was introduced in [31] and finds many applications in several areas of science, engineering and economics [1, 9, 28, 29]. A number of efficient algorithms have been