Linear Algebra and its Applications 527 (2017) 349–383

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Numerically safe Gaussian elimination with no pivoting [☆]

Victor Y. Pan [a,b,*], Liang Zhao [b]

[a] Department of Mathematics and Computer Science, Lehman College of the City University of New York, Bronx, NY 10468, USA
[b] Ph.D. Programs in Mathematics and Computer Science, The Graduate Center of the City University of New York, New York, NY 10036, USA

ARTICLE INFO

Article history:
Received 4 June 2016
Accepted 4 April 2017
Available online 8 April 2017
Submitted by A. Frommer

MSC:
15A06
15A52
15A12
65F05
65F35

Keywords:
Gaussian elimination
Pivoting
Block Gaussian elimination
Preconditioning
Random matrix algorithms
Linear systems of equations

ABSTRACT

Gaussian elimination with no pivoting and *block Gaussian elimination* are attractive alternatives to the customary but communication intensive *Gaussian elimination with partial pivoting*[1] provided that the computations proceed *safely* and *numerically safely*, that is, run into neither division by 0 nor numerical problems. Empirically, safety and numerical safety of GENP have been consistently observed in a number of papers where an input matrix was pre-processed with various structured multipliers chosen ad hoc. Our present paper provides missing formal support for this empirical observation and explains why it was elusive so far. Namely we prove that GENP is numerically unsafe for a specific class of input matrices in spite of its pre-processing with some well-known and well-tested structured multipliers, but we also prove that GENP and BGE are safe and numerically safe for the average input matrix pre-processed with any nonsingular and well-conditioned multiplier. This should embolden search for sparse and structured multipliers, and we list and test some new classes of them. We also seek randomized pre-processing that universally (that is, for all input matrices) supports (i) safe GENP and BGE with probability 1 and/or

---

[☆] Some results of this paper have been presented at the 17th Annual Conference on Computer Algebra in Scientific Computing (CASC'2014), September 10–14, 2015, Aachen, Germany (cf. [33]).

[*] Corresponding author.
E-mail addresses: victor.pan@lehman.cuny.edu (V.Y. Pan), lzhao1@gc.cuny.edu (L. Zhao).
URL: http://comet.lehman.cuny.edu/vpan/ (V.Y. Pan).

[1] Hereafter we use the acronyms *GENP*, *BGE*, and *GEPP*.

http://dx.doi.org/10.1016/j.laa.2017.04.007

(ii) numerically safe GENP and BGE with a probability close to 1. We achieve goal (i) with a Gaussian structured multiplier and goal (ii) with a Gaussian unstructured multiplier and alternatively with Gaussian structured augmentation. We consistently confirm all these formal results with our tests of GENP for benchmark inputs. We have extended our approach to other fundamental matrix computations and keep working on further extensions.

1. Introduction

1.1. *Gaussian elimination with pivoting*

The history of Gaussian elimination can be traced back some 2000 years [12]. Its modern version, GEPP (that is, Gaussian elimination with partial pivoting), is performed routinely, millions times per day around the world, being a cornerstone for computations in linear algebra [7]. For an $n \times n$ matrix, elimination involves about $\frac{2}{3}n^3$ flops, and $(n-1)n/2$ comparison are required for partial pivoting, that is, row interchange.[2] Clearly pivoting contributes a substantial share to the overall computational cost if n is small, but communication intensive pivoting takes quite a heavy toll in modern computer environment even for larger n. Pivoting interrupts the stream of arithmetic operations with foreign operations of comparison, involves book-keeping, compromises data locality, impedes parallelization of the computations, and increases communication overhead and data dependence. According to [3], "pivoting can represent more than 40% of the global factorization time for small matrices, and although the overhead decreases with the size of the matrix, it still represents 17% for a matrix of size 10,000". Because of the heavy use of GEPP, even its limited improvement is valuable.

1.2. *Contribution: random and nonrandom multipliers, safety and numerical safety*

Gaussian elimination with no pivoting (GENP) is an attractive alternative to GEPP,[3] but for some inputs can be *unsafe* or *numerically unsafe*, that is, can run into a division by 0 or numerical problems, respectively. Empirically, GENP is quite consistently safe and numerically safe [29,3,28,7] if the input matrix is pre-processed with various structured multipliers chosen ad hoc (e.g., with random circulant or SRTF multipliers),[4] but formal support for this empirical observation turned out to be elusive.

---

[2] Here and hereafter "flop" stands for "floating point arithmetic operation".

[3] Another alternative is symmetrization, but it has deficiencies for both numerical and symbolic computations: it squares the condition number of the input matrix and does not work over finite fields.

[4] SRFT is the acronym for Semisample Random Fourier Transform.