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Linear preservers of weak majorization on $\ell^1(I)^+$, when I is an infinite set

Martin Ljubenović [a,*], Dragan S. Djordjević [b,1]

[a] *Faculty of Mechanical Engineering, Department of Mathematics, University of Niš, Aleksandra Medvedeva 14, 18000 Niš, Serbia*
[b] *Faculty of Sciences and Mathematics, Department of Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia*

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The necessary and sufficient conditions that a bounded linear map on the Banach space $\ell^1(I)$, may be considered as a linear preserver of weak majorization on $\ell^1(I)^+$, where I is an arbitrary infinite set, are given. Also, we prove that the set of all linear preservers of weak majorization is closed under the norm topology.

* Corresponding author.
E-mail addresses: martinljubenovic@gmail.com (M. Ljubenović), dragan@pmf.ni.ac.rs (D.S. Djordjević).

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1. Introduction

In the finite dimensional majorization theory, there are a several equivalents of majorization relations. Basic definitions of these relations, which give a close interplay between partial sums of two real vectors have the following formulation:
Given two vectors $x, y \in \mathbb{R}^n$, we say that x is weakly majorized by y, and denote it $x \prec_w y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \qquad (k = 1, 2, \ldots, n),$$

where $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_n^{\downarrow}$ is the decreasing rearrangement of components of x. If additionally,

$$\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow},$$

then we say that x is majorized by y, and denote it $x \prec y$.

On the other hand, a matrix with non-negative real entries is called doubly substochastic, if each of its row sums and each of its column sums are less than or equal to 1. Additionally, if all row and column sums of a square matrix A are equal 1, then A is called doubly stochastic matrix. Hardy, Littlewood and Polya [23, Theorem I.1.A.3] provided that $x \prec y$ if and only if there is a doubly stochastic matrix A such that $x = Ay$. Similarly, for non-negative real vectors $x, y \in (\mathbb{R}^n)^+$, $x \prec_w y$ if and only if there is a doubly substochastic matrix A such that $x = Ay$, by [23, Theorem I.1.A.4]. We refer the reader to excellent books of majorization inequalities and their applications [10,13,23] and two standard papers [1,25].

Naturally, linear preservers as maps which preserve some property, appear in various parts of mathematics. It seems that the important branch where preserver problems are studied systematically is matrix and operator theory. For linear preserver problems in general, see [12,20,21]. In particular, linear preserver problems in the finite dimensional majorization theory have been widely studied [9,14,26,29–31].

During the last twenty years, several extensions of majorization to infinite dimensions have been raised up [27,28], and formed some applications in the operator theory such as generalisation of the well known the Schur–Horn theorem [2,3,15,16,22,24,27]. Also, there has been remarkable interest in the linear preserver problems in the operator theory as infinite dimensional variant of finite dimensional matrix case. Especially, linear preservers of extended majorization relation on the discrete Lebesgue spaces $\ell^p(I)$, $p \in [1, \infty)$, ℓ^∞ and c_0, when I is an arbitrary non-empty set are studied in [4–6,8,11]. See also, [7,17]. The extended weak majorization relation on $\ell^p(I)^+$ and the close relationship between this relation and doubly substochastic operators, as well as generalisation some of the most important results are presented in [18]. The characterisation of linear preservers of