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On the maximum rank of completions of entry pattern matrices

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ABSTRACT

In an entry pattern matrix A, all entries are indeterminates but the same indeterminate can appear in numerous positions. For a field $\mathbb{F}$, an $\mathbb{F}$-completion of A results from assigning a value from $\mathbb{F}$ to each indeterminate entry. We define the *generic* $\mathbb{F}$*-rank* of an entry pattern matrix to be its rank when considered over the function field generated over $\mathbb{F}$ by its indeterminate entries. We investigate the situation where the generic $\mathbb{F}$-rank of A is not attained by any $\mathbb{F}$-completion of A, which can occur only if the generic $\mathbb{F}$-rank exceeds the field order.

1. Introduction

An *entry pattern matrix* (EPM) A is a rectangular matrix in which each entry is an element of a specified set of independent indeterminates. A *completion* of A over a field $\mathbb{F}$ is the matrix that results from assigning a value from $\mathbb{F}$ to each indeterminate that appears as an entry of A. The set of all $\mathbb{F}$-completions of A is a vector space over $\mathbb{F}$, whose dimension is equal to the number of distinct indeterminates appearing in A.

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The concept of an entry pattern matrix was introduced by Huang and Zhan in 2015 [3]. These authors note that several important classes of matrices, including symmetric, circulant, Toeplitz and Hankel matrices, have defining properties that can be expressed in terms of entry patterns. They investigate the particular case of entry pattern matrices whose real completions all have real spectrum, and they propose a general study of matrix-theoretic properties of entry patterns and their completions. This introduces a new focus to the subject of matrices with indeterminate entries. Other classes of such objects that have attracted recent attention include partial matrices [5] and affine column independent (ACI) matrices [1,2].

This article is concerned with rank bounds for square entry pattern matrices. The basic theme is the determination of the maximum rank of completions of a given EPM over a specified field $\mathbb{F}$. An upper bound for this number is given by the *generic* $\mathbb{F}$*-rank*, which is the rank of the EPM when considered as a matrix written over the function field generated over $\mathbb{F}$ by the indeterminate entries. Provided that the field $\mathbb{F}$ has high enough order, we will see that the generic $\mathbb{F}$-rank can be attained by some $\mathbb{F}$-completion and so the generic and maximum ranks coincide. However, examples exist over small finite fields of entry pattern matrices whose generic rank exceeds their maximum rank. We will show that this can occur only if the matrix order exceeds the field order, and we will say that a finite field $\mathbb{F}$ is *EPM-rank-tight* if there exists a generically nonsingular square EPM of order $|\mathbb{F}| + 1$ whose $\mathbb{F}$-completions are all singular. We present some explicit constructions to show that the property of EPM-rank-tightness is preserved by finite extension of fields, and to demonstrate that certain fields of prime order possess this property. For a prime power q, we write $\mathbb{F}_q$ for the finite field with q elements.

Given a finite set $S = \{x_1, x_2, \cdots, x_k\}$, we denote by $M_{m\times n}(S)$, or $M_{m\times n}(x_1, \cdots, x_k)$ the set of $m \times n$ matrices whose entries are elements of S. A matrix $A \in M_{m\times n}(S)$ may also be written as $A(x_1, \cdots, x_k)$ and is referred to as an entry pattern matrix with indeterminates $x_1, \ldots, x_k$. The $\mathbb{F}$-*pattern class* of an entry pattern matrix A, denoted by $C_{\mathbb{F}}(A)$, is the linear space consisting of all matrices that can be obtained by specifying the values of the indeterminates of A as elements in the field $\mathbb{F}$. Its dimension is the number of distinct indeterminates in A. An element of $C_{\mathbb{F}}(A)$ is called an $\mathbb{F}$-completion of A. The matrix in $C_{\mathbb{F}}(A)$ which results from setting $x_1 = a_1, \cdots, x_k = a_k$ is written as $A(a_1, \cdots, a_k)$.

Definition 1.1. The *maximum* $\mathbb{F}$*-rank* of A, denoted by $\mathrm{m}_{\mathbb{F}}\text{-rank}(A)$, is the maximum rank of all of $\mathbb{F}$-completions of A.

$$\mathrm{m}_{\mathbb{F}}\text{-rank}(A(x_1, \ldots, x_k)) = \max_{a_1,\ldots,a_k\in\mathbb{F}} \mathrm{rank}(A(a_1, \ldots a_k)).$$

Definition 1.2. The *generic* $\mathbb{F}$*-rank* of A, denoted by $\mathrm{g}_{\mathbb{F}}\text{-rank}(A)$, is the rank of A when considered as a matrix in $M_{m\times n}(\overline{\mathbb{F}})$, where $\overline{\mathbb{F}}$ is the function field generated over $\mathbb{F}$ by the indeterminate entries of A. We say that a square EPM $A \in M_n(S)$ is $\overline{\mathbb{F}}$-nonsingular if $\mathrm{g}_{\mathbb{F}}\text{-rank}(A) = n$.